UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 9:00 am to 11:00 am

PAPER 67

ADVANCED QUANTUM INFORMATION THEORY

*Attempt no more than **TWO** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) For a system of qudits, let A_X and B_Y be operators that act non-trivially only on qudits in the sets X and Y. $A_X(t)$ denotes the time-evolution of A_X in the Heisenberg picture.

Let Z range over k-element subsets of the qudits, and let $H = \sum_Z h_Z$ be a k-local Hamiltonian where h_Z acts non-trivially only on the subset Z.

Assume that H satisfies the Lieb-Robinson bound:

$$\|[A_X(t), B_Y]\| \leqslant 2\,\|A_X\|\,\|B_Y\|\,\min(|X|,|Y|)\,e^{-\mu d(X,Y)}\big(e^{2kst}-1\big),$$

where $d(X,Y)$ denotes interaction distance, and μ and s are constants.

You may use without proof the fact that, for any operator $O_{AB} \in \mathcal{B}(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$,

$$\mathrm{Tr}_B[O_{AB}] \otimes \mathbb{1}_B = d_B \int \mathrm{d}U(\mathbb{1} \otimes U) O_{AB} (\mathbb{1} \otimes U^\dagger)$$

where the integral is over the Haar measure for the unitary group $SU(d_B)$, normalised such that $\int \mathrm{d}U = 1$.

Prove that there exists an operator $A_{X(l)}$ acting non-trivially only on qudits within the subset $X(l) = \{i : d(i,X) \leqslant vt + l\}$, such that

$$\big\|A_X(t) - A_{X(l)}(t)\big\| \leqslant \mu vt|X|\|A_X\|e^{-\mu l/2},$$

where $v > 0$ is a constant, and find an expression for v in terms of the parameters of the system.

(b) Consider a 1-dimensional chain of qudits with nearest-neighbour Hamiltonian $H = \sum_i (\mathbb{1} - P_{i,i+1})$, where $P_{i,i+1}$ are projectors that act non-trivially only on qudits i and $i+1$.

Assume that H has a unique ground state $|\psi_0\rangle$, spectral gap $\Delta > 0$, and is frustration-free, i.e. $\forall i : P_{i,i+1}|\psi_0\rangle = |\psi_0\rangle$.

Let $P_{\text{odd}} := \prod_i P_{2i-1,2i}$, $P_{\text{even}} := \prod_i P_{2i,2i+1}$, and $K := P_{\text{odd}} P_{\text{even}}$. The Detectability Lemma states that:

$$\|K|_{\operatorname{supp} H}\| \leqslant \frac{1}{\left(\frac{\Delta}{2}+1\right)^{1/3}}.$$

(i) Using the Detectability Lemma, or otherwise, show that $\|K^m - |\psi_0\rangle\langle\psi_0|\| \leqslant O(e^{-\alpha m})$ for some constant α, and find an expression for α in terms of the parameters of the Hamiltonian.

(ii) Let A_x be an operator that acts non-trivially only on qudit x of the chain. Construct an operator $A_{x(m)}$ acting non-trivially *only* on qudits within distance m of x, such that

$$\big\|(|\psi_0\rangle\langle\psi_0|)A_x|\psi_0\rangle - A_{x(m)}|\psi_0\rangle\big\| \leqslant O\left(\|A_x\|\,e^{-\alpha m}\right)$$

for the same constant α as in part (b)(i).

Hint: it may help to sketch a diagram of K^m *acting on* $A_x |\psi_0\rangle$.

(iii) Let A_x and B_y be observables on qudits x and y, respectively. Prove that, in the ground state $|\psi_0\rangle$, the connected correlation function between A_x and B_y decays exponentially with distance between qudits x and y

Give an explicit expression for the rate of decay in terms of the spectral gap.

2

Consider a system of qubits whose overall Hilbert space is $\mathcal{H} = (\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes T}$. The subscripts $1 \dots T$ label the T qubits in the $(\mathbb{C}^2)^{\otimes T}$ part of the Hilbert space, so that e.g. $|x\rangle\langle x|_{i,j}$ acts non-trivially only on qubits i and j of $(\mathbb{C}^2)^{\otimes T}$.

Consider a quantum circuit on n qubits consisting of T quantum gates described by unitaries U_t, $t = 1 \dots T$, where each unitary acts non-trivially on at most two qubits.

Define the following Hamiltonians acting on $\mathcal{H}$:

$$\begin{aligned}
H_{\text{init}} &= |1\rangle\langle 1|_1\,, \qquad H_{\text{final}} = \sum_{t=1}^{T} H_t \\
H_1 &= \frac{1}{2}\mathbb{1} \otimes (|00\rangle\langle 00|_{1,2} + |10\rangle\langle 10|_{1,2}) \\
&\quad - \frac{1}{2}U_1 \otimes |10\rangle\langle 00|_{1,2} - \frac{1}{2}U_1^\dagger \otimes |00\rangle\langle 10|_{1,2} \\
H_T &= \frac{1}{2}\mathbb{1} \otimes (|10\rangle\langle 10|_{T-1,T} + |11\rangle\langle 11|_{T-1,T}) \\
&\quad - \frac{1}{2}U_T \otimes |11\rangle\langle 10|_{T-1,T} - \frac{1}{2}U_T^\dagger \otimes |10\rangle\langle 11|_{T-1,T}
\end{aligned}$$

For $1 < t < T$:

$$\begin{aligned}
H_t &= \frac{1}{2}\mathbb{1} \otimes \Big(|100\rangle\langle 100|_{t-1,t,t+1} + |110\rangle\langle 110|_{t-1,t,t+1}\Big) \\
&\quad - \frac{1}{2}U_t \otimes |110\rangle\langle 100|_{t-1,t,t+1} - \frac{1}{2}U_t^\dagger \otimes |100\rangle\langle 110|_{t-1,t,t+1}
\end{aligned}$$

Let $\mathcal{L}$ be the subspace $\mathcal{L} = \operatorname{span}\Big\{|\psi_t\rangle\,|1\rangle^{\otimes t}\,|0\rangle^{\otimes T-t}\Big\}_t \subset \mathcal{H}$, where $|\psi_t\rangle = \prod_{i=1}^{t} U_t\,|\psi_0\rangle$ for some fixed state $|\psi_0\rangle$.

For any operator X and subspace S, $X|_S$ denotes the restriction of X to S.

(a) Show that $\mathcal{L}$ is an invariant subspace of both H_{init} and H_{final}.

(b) In this part of the question, you may use without proof the fact that the matrix $E = \frac{1}{2}\sum_{i=0}^{N}(|i\rangle - |i+1\rangle)(\langle i| - \langle i+1|)$ has eigenvalues $\lambda_k = 1 - \cos q_k$ where $q_k = \frac{\pi k}{N+1}$, $k = 0, \dots, N$, and that the eigenvector corresponding to λ_0 is $\frac{1}{\sqrt{N+1}}\sum_{i=0}^{N}|i\rangle$.
You may also use without proof the Gershgorin Disc Theorem, which states that for any matrix M, the eigenvalues of M are contained in the union of the discs D_i in the complex plane:

$$D_i = \Big\{z : |z - M_{ii}| \leqslant \sum_{j \neq i} |M_{ij}|\Big\},$$

where M_{ij} denotes the i,jth element of M. Furthermore, a disc which does not intersect with any other discs contains exactly one eigenvalue.

(i) Find the eigenvalues of $H_{\text{init}}|_{\mathcal{L}}$.

(ii) Prove that the spectral gap $\Delta(H_{\text{final}}|_{\mathcal{L}})$ of $H_{\text{final}}|_{\mathcal{L}}$ satisfies $\Delta(H_{\text{final}}|_{\mathcal{L}}) \geqslant \Omega\left(\frac{1}{T^2}\right)$.

(iii) Prove that $H_{\text{final}}|_{\mathcal{L}}$ has a ground state that is a computational history state for the quantum circuit.

(iv) Let $H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}$ for $s \in [0,1]$. Prove that the spectral gap $\Delta(H(s)|_{\mathcal{L}}) \geqslant \frac{1}{3}$ for $0 \leqslant s \leqslant \frac{1}{3}$.

(c) Let $|\phi(0)\rangle$ and $|\varphi\rangle$ be ground states of $H_{\text{init}}|_{\mathcal{L}}$ and $H_{\text{final}}|_{\mathcal{L}}$, respectively.

The Adiabatic Theorem says that if the system is prepared in the state $|\phi(0)\rangle$, and the Hamiltonian $H(s) = (1-s)H_{\text{init}} + sH_{\text{final}}$ is adiabatically varied from $s = 0$ to 1 at a constant rate, the final state of the system $|\phi(1)\rangle$ will be ϵ-close in trace distance to $|\varphi\rangle$, i.e.

$$\big\| \, |\phi(1)\rangle\langle\phi(1)| - |\varphi\rangle\langle\varphi| \, \big\|_1 \leqslant \epsilon,$$

providing that the total time τ for the adiabatic evolution satisfies

$$\tau \geqslant \Omega\left(\frac{\|H_{\text{final}}|_{\mathcal{L}} - H_{\text{init}}|_{\mathcal{L}}\|^2}{\epsilon \min_{s\in[0,1]} \Delta(H(s)|_{\mathcal{L}})^3}\right).$$

You may assume without proof that the spectral gap $\Delta(H(s)|_{\mathcal{L}}) \geqslant \Omega\left(\frac{1}{T^2}\right)$ for $\frac{1}{3} < s \leqslant 1$.

Using the results of parts (a) and (b), or otherwise, prove that a state ϵ-close in trace distance to a computational history state for the quantum circuit can be prepared adiabatically in a time that scales polynomially in the size of the circuit.

3

Throughout this question, you may assume without proof that any classical computation can be carried out in the quantum circuit model with at most polynomial overhead.

You may also assume without proof that any unitary on a constant number of qudits can be implemented by a quantum circuit using a constant number of gates.

You may use without proof the following Chernov bound. Let y_i, $i = 1 \ldots n$, be n independent Bernoulli (0/1-valued) random variables with $\Pr(1) = p_*$. (I.e. a biased coin with $\Pr(\text{heads}) = \Pr(1) = p_*$ is tossed n times, each outcome is independent, and $y_i \in \{0, 1\}$ is the outcome of the ith coin toss.) Then

$$\Pr\left(\frac{\sum_{i=1}^n y_i}{n} - p_* > \frac{1}{2} - p_*\right) < e^{-(1-2p_*)n} \qquad \text{if } p_* < \frac{1}{2},$$
$$\Pr\left(p_* - \frac{\sum_{i=1}^n y_i}{n} > p_* - \frac{1}{2}\right) < e^{-(2p_*-1)n} \qquad \text{if } p_* > \frac{1}{2}.$$

(a) Define the complexity class QMA.

(b) Prove rigorously that defining QMA with success probability $O\left(1 - \frac{1}{\text{poly}(n)}\right)$, where n is the problem instance size, gives rise to the same complexity class as QMA defined with success probability $\frac{2}{3}$.

(c) Define the Local Hamiltonian problem.

(d) Prove that the Local Hamiltonian problem is contained in QMA.

4

(a) State and prove Kitaev's Geometrical Lemma, defining all quantities involved.

The function $w(t)$ with Fourier transform $\hat{w}(E)$ satisfies:

$$w(t) \geqslant 0, \qquad \hat{w}(E) = 0 \text{ for } E \geqslant \Delta, \qquad \hat{w}(0) = 1, \quad \text{and}$$
$$\forall T > 0 : \int_T^\infty w(t)\mathrm{d}t = O\left((\ln \beta T)^2\, e^{-\beta T/(\ln \beta T)^2}\right) \text{ for some constant } \beta > 0.$$

For a system of qudits, let $H = \sum_Z h_Z$ be a k-local Hamiltonian, where Z in the summation ranges over k-element subsets of the qudits and h_Z acts non-trivially only on the subset Z.

For the remainder of this question, you may assume that H has a unique ground state $|\phi_0\rangle$, eigenstates $|\phi_i\rangle$, and spectral gap $\Delta > 0$.

(b) Using the Fourier transform and filtering technique, prove that H can be rewritten as $H = \sum_Z g^{(Z)}$ with $[g^{(Z)}, P_0] = 0$, where $P_0 := |\phi_0\rangle\langle\phi_0|$.

(The operators $g^{(Z)}$ may act on all the qudits in the system.)

(c) Let

$$g^{(Z,s)} := \int_{-\infty}^{\infty} w(t) \left(e^{itH_s}\, h_Z\, e^{-itH_s} - e^{itH_{s-1}}\, h_Z\, e^{-itH_{s-1}}\right) \mathrm{d}t,$$

for $s \geqslant 1$, where $H_s := \sum\limits_{Y\,:\,d(Z,Y)\leqslant s} h_Y$ and $d(Z, Y)$ denotes interaction distance, so that $H_0 = h_Z$ and $H_D = H$.

Show that $g^{(Z)} = h_Z + \sum_{s=1}^{D} g^{(Z,s)}$ for suitably chosen $g^{(Z)}$ from part (b).

(d) A frustration-free Hamiltonian is a Hamiltonian whose overall ground state is also the ground state of each local term considered separately.

Is the Hamiltonian $H = \sum_Z g^{(Z)}$ with local terms $g^{(Z)}$ defined as in part (c) frustration-free? If so, why? If not, why not?

END OF PAPER