UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 1:30 pm to 4:30 pm

PAPER 66

QUANTUM INFORMATION THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

i. What is a Stinespring representation of a completely positive map?

ii. Suppose ρ_{Q} is a state with eigendecomposition $\rho_{\mathsf{Q}} = \sum_{0 \leqslant i < d} \lambda_i |\alpha_i\rangle\langle\alpha_i|$. Write down the von Neumann entropy $S(\rho_{\mathsf{Q}})$ of ρ_{Q} as a function of the $\{\lambda_i : 0 \leqslant i < d\}$.

iii. Define the conditional entropy $H(\mathsf{A}|\mathsf{B})_\rho$ and coherent information $I(\mathsf{A}\rangle\mathsf{B})_\rho$ in terms of von Neumann entropies.

iv. Why is it true that for any *pure* state ψ_{QR} of a bipartite system QR, $H(\mathsf{Q})_\psi = H(\mathsf{R})_\psi$?

v. The data processing inequality for the coherent information states that if ρ_{AB} is a state and $\mathcal{N}^{\mathsf{B}'\leftarrow\mathsf{B}}$ is an operation and $\sigma_{\mathsf{AB}'} = \mathcal{N}^{\mathsf{B}'\leftarrow\mathsf{B}}\rho_{\mathsf{AB}}$, then

$$I(\mathsf{A}\rangle\mathsf{B})_\rho \geqslant I(\mathsf{A}\rangle\mathsf{B}')_\sigma.$$

Prove this data processing inequality, stating clearly any results you use.

vi. For any state ρ_{ABC}, show that

$$H(\mathsf{A})_\rho + H(\mathsf{B})_\rho \leqslant H(\mathsf{AC})_\rho + H(\mathsf{BC})_\rho.$$

2

In both parts of this question we are assuming that initially Alice possesses a qubit A and Bob a qubit B which are in the state $\phi^+_{\mathsf{AB}} = |\phi^+\rangle\langle\phi^+|_{\mathsf{AB}}$ where $|\phi^+\rangle_{\mathsf{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathsf{A}} \otimes |0\rangle_{\mathsf{B}} + |1\rangle_{\mathsf{A}} \otimes |1\rangle_{\mathsf{B}})$.

i. Describe in detail how Alice can send two bits of classical information to Bob by transmission of a single qubit, showing why the protocol you describe works.

ii. Suppose Alice has some real parameter θ in mind, which is unknown to Bob. Describe a protocol involving only local operations and the communication of a *single bit* (i.e. a message taking a value in $\{0, 1\}$) from Alice to Bob whereby Bob's qubit ends up in the state $|\alpha_\theta\rangle\langle\alpha_\theta|$ where $|\alpha_\theta\rangle := \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$.

3

In the following question X is an hermitian operator on a d-dimensional Hilbert space $\mathcal{H}$ with an eigendecomposition $X = \sum_{i=1}^{d} \lambda_i |\alpha_i\rangle\langle\alpha_i|$.

i. Write down the positive part of X, X_+, the negative part of X, X_-, in terms of the given eigendecomposition.

Define the operator $|X|$ and show that $|X| = X_+ + X_-$.

ii. Define the trace norm $\|X\|_1$ and state the Holevo-Helstrom theorem.

iii. Show that

$$\|X\|_1 = \max\{\mathrm{Tr} XT : -I \leqslant T \leqslant I\},$$

where I denotes the identity operator on $\mathcal{H}$. Use this to prove the Holevo-Helstrom theorem.

iv. For states ρ and σ, define the trace distance $D(\rho, \sigma)$ and fidelity $F(\rho, \sigma)$.

v. If $\{|\psi_i\rangle : i = 1, \ldots, d\}$ is any orthonormal basis for $\mathcal{H}$, show that

$$\|X\|_1 \geqslant \sum_{i=1}^{d} |\langle\psi_i|X|\psi_i\rangle|,$$

stating clearly any results you use. Hence show that for an arbitrary state ρ and pure state $|\psi\rangle\langle\psi|$

$$D(\rho, |\psi\rangle\langle\psi|) \geqslant 1 - F(\rho, |\psi\rangle\langle\psi|)^2.$$

4

i. For a random variable X taking values in a finite set $\mathcal{A}_X$ with distribution P_X:

Define the entropy of X, $H(X)$.

What is an ϵ-sufficient subset for X?

ii. Let Z and Z_i for $i \in \{1, 2, \ldots\}$ be random variables, each taking values in a finite set $\mathcal{A}_Z$, and independently and identically distributed according to P_Z.

For $n \in \{1, 2, \ldots\}$, let $Z^{(n)} = (Z_1, Z_2, \ldots, Z_n)$.

Define the δ-typical subset $T_\delta^n(P_Z) \subseteq \mathcal{A}_Z^n$ and, stating clearly any results you use, show that:

(a) For all $\delta > 0$: $\lim_{n\to\infty} \Pr(Z^{(n)} \in T_\delta^n(P_Z)) = 1$.

(b) $|T_\delta^n(P_Z)| \leqslant 2^{(H(Z)+\delta)n}$.

(c) For any $\epsilon < 1$, $\delta > 0$ and all sufficiently large n, if S_n is an ϵ-sufficient set for $Z^{(n)}$ then

$$|S_n| \geqslant \frac{1-\epsilon}{2} 2^{(H(Z)-\delta)n}.$$

iii. A single use of a k-ary erasure channel takes an input symbol X in $\mathcal{A}_X := \{0, 1, \ldots, k-1\}$, and produces an output symbol Y in $\mathcal{A}_Y := \{\mathbf{e}, 0, 1, \ldots, k-1\}$, with the conditional distribution

$$N_{Y|X}(y|x) = \begin{cases} 1-f & \text{if } y = x, \\ f & \text{if } y = \mathbf{e}, \\ 0 & \text{otherwise.} \end{cases}$$

Use Shannon's formula to show that the capacity of the discrete memoryless channel $N_{Y|X}$ is $(1-f)\log k$.

5

i. If X is a random variable taking values in $\{1, \ldots, k\}$ with distribution P_X and $P_X(1) = t$ show that the entropy of X, $H(X)$, satisfies

$$H(X) \leqslant h(t) + (1-t)\log_2(k-1)$$

where $h(t) := (1-t)\log_2 \frac{1}{1-t} + t\log_2 \frac{1}{t}$ is the binary entropy function.

ii. Let ρ be a density operator on a d-dimensional Hilbert space $\mathcal{H}$, let $\{|\psi_i\rangle : i = 1, \ldots, d\}$ be an orthonormal basis for $\mathcal{H}$, and let

$$\rho' = \sum_{i=1}^{d} |\psi_i\rangle\langle\psi_i|\rho|\psi_i\rangle\langle\psi_i|.$$

Show that $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\rho')$ and use Klein's inequality to show that $S(\rho') \geqslant S(\rho)$.

iii. Let $|\rho\rangle\langle\rho|_{\mathsf{QR}}$ be a purification of a state ρ_{Q}, where the systems Q and R both have Hilbert spaces of dimension d. Let $\sigma_{\mathsf{QR}} := \mathcal{N}^{\mathsf{Q}\leftarrow\mathsf{Q}} \otimes \mathbf{id}^{\mathsf{R}\leftarrow\mathsf{R}}|\rho\rangle\langle\rho|_{\mathsf{QR}}$ and $f^2 := \langle\rho|_{\mathsf{QR}}\sigma_{\mathsf{QR}}|\rho\rangle_{\mathsf{QR}}$. Using the previous parts of the question, show that

$$S(\sigma_{\mathsf{QR}}) \leqslant h(f^2) + (1-f^2)\log_2(d^2-1).$$

END OF PAPER