UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 9:00 am to 11:00 am

PAPER 64

DYNAMICS OF ASTROPHYSICAL DISCS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Steady state disk emission

The diffusion equation governing the evolution of a Keplerian accretion disk around a compact object of mass M_* is

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r}\partial_r \mathcal{F},$$

where the mass flux $\mathcal{F}$ is defined by

$$\mathcal{F} = 6\pi r^{1/2}\partial_r \left(r^{1/2}\overline{\nu}\Sigma\right),$$

with Σ the disk surface density and $\overline{\nu}$ the mean turbulent viscosity. The viscous torque vanishes at the radius of the central object $r = r_*$, and the disk is supplied with mass at its outer radius $r = r_{\rm out} \gg r_*$ at a rate $\dot{M}$.

(a) Calculate the steady state disk structure $\Sigma = \Sigma(r)$.

(b) Assume the disk is in steady state energy balance so that cooling balances heating: $\mathcal{C} = \mathcal{H}$. The disk surfaces radiate as blackbodies so that $\mathcal{C} = 2\sigma T^4$, where σ is the Stefan-Boltzmann constant and T is the effective temperature. The heating rate is given by $\mathcal{H} = \frac{9}{4}\overline{\nu}\Sigma\Omega^2$, where $\Omega = \Omega(r)$ is the orbital frequency. Show that

$$T = T_{\rm in}\left(\frac{r_*}{r}\right)^{3/4}\left(1 - \sqrt{\frac{r_*}{r}}\right)^{1/4},$$

where $T_{\rm in} = (3GM_*\dot{M}/8\pi\sigma r_*^3)^{1/4}$.

Consider a white dwarf and a neutron star of roughly the same mass and accreting at the same rate. The white dwarf has radius $\sim 10^4$ km, while the neutron star has a radius of ~ 10 km. How much hotter is the disk around the neutron star and how will this influence its emitted spectrum?

(c) The spectral energy flux of the disk is given by

$$F_\nu \propto \nu^3 \int_{r_*}^{r_{\rm out}} \frac{r}{e^{h\nu/kT} - 1}\, dr,$$

where ν is the frequency of the radiation, and h and k are constants.

Show that $F_\nu \propto \nu^2$ in the low frequency limit $\nu \ll kT_{\rm out}/h$, where $T_{\rm out}$ is the temperature at the outer boundary of the disk. Which region of the disk is primarily emitting in this range?

Next show that $F_\nu \propto \nu^{1/3}$ for intermediate frequencies $kT_{\rm out}/h \ll \nu \ll kT_{\rm in}/h$. You may assume for this calculation that $T \approx T_{\rm in}(r_*/r)^{3/4}$.

(In both parts of the question the constant of proportionality involves a dimensionless integral you need not evaluate)

(d) The luminosity of an annulus of disk between $r = r_1$ and $r = r_2$ is

$$L = \int_{r_1}^{r_2} 2\pi r \mathcal{H}\, dr.$$

Show that the total luminosity of the disk is approximately $GM_*\dot{M}/(2r_*)$. Next show that this is only half the potential energy liberated by accretion of material to the radius $r = r_*$. What happens to the remaining energy?

(e) The disk connects to the central star via a boundary layer of thickness $\sim H$, the disk's local scale height. Assuming the boundary layer emits as a blackbody, show that the effective temperature of the boundary layer is

$$T_{BL} \sim \left(\frac{r_*}{H}\right)^{1/4} T_{\rm in} \,.$$

What can you say about the boundary layer's emission in comparison to the disk's?

2 Compressible dynamics and gravitational instability

The equations governing compressible gas in the shearing sheet are

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\Sigma}\nabla P - 2\Omega\,\mathbf{e}_z\times\mathbf{u} - \nabla\Phi_t,$$
$$\partial_t\Sigma + \mathbf{u}\cdot\nabla\Sigma = -\Sigma\nabla\cdot\mathbf{u},$$

where $\mathbf{u}$ is velocity, Σ is surface density, the tidal potential is $\Phi_t = -\frac{3}{2}\Omega^2x^2$, and Ω is the orbital frequency of the sheet. The gas is assumed to be isothermal so the pressure is $P = c_s^2\Sigma$, where c_s is the (constant) isothermal sound speed.

(a) Obtain the following evolution equation for the total energy,

$$\partial_t\left(\tfrac{1}{2}\Sigma\,\mathbf{u}^2 + c_s^2\Sigma\,\ln\Sigma + \Sigma\Phi_t\right) + \nabla\cdot\mathbf{F} = 0,$$

where $\mathbf{F}$ is the energy flux, the form of which you should give.

(You may need the vector identity: $\mathbf{A}\times\nabla\times\mathbf{A} = \frac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A}) - \mathbf{A}\cdot\nabla\mathbf{A}$.*)*

(b) A disk is gravitationally unstable if $Q < 1$, where

$$Q = \frac{c_s\,\kappa}{\pi\,G\,\Sigma_0},$$

and κ is the epicyclic frequency. Describe the different physical processes competing for the stability of the disk and how they are represented in the Q parameter. Give an approximate form of the stability criterion in terms of the scale height (H) and the masses of the disk (M_D) and the central star (M_*).

A popular model for the protosolar nebula sets

$$\Sigma_0 \sim 10^3\left(\frac{r}{1\,\mathrm{AU}}\right)^{-3/2}\,\mathrm{g\,cm^{-2}},$$

where 1 AU $\sim 10^{13}$ cm. If the disk aspect ratio is a constant and equal to 0.1, estimate the value of Q, and hence determine the disk's stability, at 1 AU. At what radius would the protosolar nebula be unstable? Discuss the likelihood that planets in the solar system formed via gravitational instability in the disk.

(Assume that $G \sim 7\times10^{-8}$ cm^3 g^{-1} s^{-2}.)

(c) The gravitational potential of a disk is obtained from Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho,$$

where ρ is the disk's volumetric mass density.

A razor-thin two-dimensional disk is modelled with $\rho = \Sigma\,\delta(z)$, where $\delta(z)$ is the Dirac delta function. Consider a small density perturbation $\Sigma' = \Sigma_0 e^{ikx}$ upon an otherwise homogeneous disk, where Σ_0 is a constant. Compute Φ', the corresponding perturbation in the gravitational potential.

(d) Next consider a fully three-dimensional disk of density ρ. Show, using index notation or otherwise, that the self-gravitational force in the momentum equation $-\rho\nabla\Phi$ may be written as the divergence of a stress $-\nabla\cdot\mathbf{T}$, where

$$T_{ij} = \rho\left(g_i\,g_j - \tfrac{1}{2}\delta_{ij}\,g^2\right),$$

and δ_{ij} is the Kronecker delta, and the 'gravitational velocity' is defined by $\mathbf{g} = \nabla\Phi/\sqrt{4\pi G\rho}$.

Which term in the above do you think contributes to angular momentum transport and why? What is the effect of the other term?

3 The magnetorotational instability in a radially stratified disk

Consider an accretion disk with a radially varying thermal structure. A local patch of the disk may be represented in the shearing sheet by the following set of equations

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho_0}\nabla\psi - 2\Omega\,\mathbf{e}_z\times\mathbf{u} + 3\Omega^2 x\,\mathbf{e}_x - N^2\,\theta\,\mathbf{e}_x + \frac{1}{4\pi\rho_0}\mathbf{B}\cdot\nabla\mathbf{B},$$

$$\partial_t\mathbf{B} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u}, \qquad \partial_t\theta + \mathbf{u}\cdot\nabla\theta = u_x,$$

$$\nabla\cdot\mathbf{u} = 0, \qquad \nabla\cdot\mathbf{B} = 0,$$

where $\mathbf{u}$, $\mathbf{B}$, ψ, and ρ_0 are the velocity, magnetic field, total pressure, and density, respectively. In addition, θ is the 'potential temperature' perturbation. Finally, Ω is the orbital frequency of the shearing sheet and N^2 is the squared radial buoyancy frequency of the gas. Both ρ_0 and N^2 are constants.

(a) The disk supports the equilibrium state: $\mathbf{u} = -\frac{3}{2}\Omega\, x\,\mathbf{e}_y$, $\mathbf{B} = B_0\,\mathbf{e}_z$, with ψ a constant and $\theta = 0$. Consider general perturbations to this state and write down the 9 linearised equations governing their evolution.

(b) Suppose the perturbations are independent of x and y and proportional to $\mathrm{e}^{\sigma t + ikz}$. First show that the pressure and vertical components of the velocity and magnetic field perturbations are zero. Subsequently derive the dispersion relation

$$\sigma^4 + \left(2v_A^2k^2 + N^2 + \Omega^2\right)\sigma^2 + v_A^2k^2\left(v_A^2k^2 + N^2 - 3\Omega^2\right) = 0,$$

where $v_A^2 = B_0^2/(4\pi\rho_0)$.

(c) Consider the case when $B_0 = 0$, and establish the Hoiland-Solberg instability criterion, $N^2 + \Omega^2 < 0$.

The buoyancy frequency is given by $N^2 \propto -\rho^{-1}\partial_r P\,\partial_r S$, evaluated at the radius of the shearing sheet. Here P and S are the equilibrium gas pressure and dimensionless entropy of the global disk. Both quantities are assumed to decrease with radius. Give an order of magnitude estimate for N^2 and comment on the hydrodynamical stability of the disk.

(d) Consider now $B_0 \neq 0$. Establish the instability criterion in this case, then show that the maximum growth rate occurs when

$$v_A^2k^2 = \Omega^2 - \frac{(N^2+\Omega^2)^2}{16\Omega^2}.$$

What condition must hold for the above expression to make sense? Finally, find the maximum growth rate and comment on the influence of the thermal radial structure on the magnetorotational instability.

END OF PAPER