UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 1:30 pm to 4:30 pm

PAPER 63

PLANETARY SYSTEM DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A nearly edge-on axisymmetric circumstellar disk is observed with a mid-plane inclined $I_{\rm d} \approx 87^\circ$ to the sky-plane. An orthonormal astrocentric reference frame $[\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}]$ is defined with $\hat{\mathbf{X}}$ and $\hat{\mathbf{Y}}$ in the sky-plane, $\hat{\mathbf{X}}$ pointing toward North and $\hat{\mathbf{Z}}$ toward Earth. The disk's ascending node through the sky-plane (in which disk material approaches us) lies at a position angle $\Omega_{\rm d} = 45^\circ$ from North. Another orthonormal astrocentric reference frame $[\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}]$ is defined with $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in the disk mid-plane with $\hat{\mathbf{x}}$ in the sky-plane in the direction of the disk's ascending node, at which location material moves in the $\hat{\mathbf{y}}$ direction. Provide an annotated sketch of this geometry, and determine the transformation matrix T that converts between locations in the two reference frames $\mathbf{X}$ and $\mathbf{x}$, respectively, such that $\mathbf{X} = T\,\mathbf{x}$.

A planet is also observed to orbit the star with an inclination to the sky-plane of $I_{\rm p} \approx 88^\circ$. The planet's ascending node through the sky-plane lies at a position angle $\Omega_{\rm p} = 49^\circ$ from North. The planet's ascending node through the disk mid-plane (that in which motion is in the positive $\hat{\mathbf{z}}$ direction) is in the direction $\hat{\mathbf{x}}'$ at an angle $\Omega_{\rm m}$ from the $\hat{\mathbf{x}}$ direction. The orthonormal astrocentric reference frame $[\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}']$ has $\hat{\mathbf{y}}'$ in the planet's orbital plane oriented in the direction of motion at the ascending node. Provide an updated sketch of this geometry.

There are two ways of applying successive rotations to convert between the $[\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}]$ and $[\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}']$ reference frames. Describe how this can be used to determine both $\Omega_{\rm m}$ and the inclination of the planet's orbit to the disk mid-plane $I_{\rm m}$ as a function of $I_{\rm d}$, $I_{\rm p}$, $\Omega_{\rm p}$ and $\Omega_{\rm d}$, defining any additional angles needed for the transformation.

Hence, or otherwise, show that

$$\begin{aligned} \cos I_{\rm m} &= \cos I_{\rm d} \cos I_{\rm p} + \sin I_{\rm d} \sin I_{\rm p} \cos(\Omega_{\rm p} - \Omega_{\rm d}), \\ \tan \Omega_{\rm m} &= \frac{\sin I_{\rm p} \sin(\Omega_{\rm p} - \Omega_{\rm d})}{\cos I_{\rm d} \sin I_{\rm p} \cos(\Omega_{\rm p} - \Omega_{\rm d}) - \sin I_{\rm d} \cos I_{\rm p}}. \end{aligned}$$

Find an expression for $I_{\rm m}$ to second order in the small quantities for the system in this example, and sketch $I_{\rm m}$ as a function of $I_{\rm p}$ for $85^\circ < I_{\rm p} < 95^\circ$, quantifying where possible.

Disk material is spread over a large range of radii from $r_{\rm in}$ to $r_{\rm out}$, and the orbit of the planet is significantly interior to the disk. Describe how the orbital planes of disk material at different radii would be expected to evolve due to the gravity of the planet on a plot of $I_{\rm m} \cos \Omega_{\rm m}$ versus $I_{\rm m} \sin \Omega_{\rm m}$.

Hence explain why the planet causes the disk to appear warped during some epoch of its evolution.

2

A dust particle with radiation pressure coefficient β orbits a star of mass $M_\star$ near a planet of mass $M_{\rm p} \ll M_\star$ that is on a circular orbit at a semimajor axis $a_{\rm p}$. The particle is on a circular orbit at a semimajor axis $a > a_{\rm p}$. Determine the semimajor axis at which the particle orbits the star j times for every $j+1$ planet orbits, expressed using $\epsilon \equiv (a-a_{\rm p})/a_{\rm p}$.

Hence show that the planet's first order mean motion resonances for this particle are interior to the planet's orbit for

$$j > \frac{\sqrt{1-\beta}}{1-\sqrt{1-\beta}}.$$

From an initially coplanar circular orbit at a semimajor axis a_1, where $\epsilon_1 \ll 1$, the particle undergoes a single encounter with the planet. Show that to first order in small quantities the velocity at which the particle encounters the planet in the frame rotating with the planet is $v_\infty \approx v_{\rm p}[1-(1-\epsilon_1/2)\sqrt{1-\beta}]$, where $v_{\rm p}$ is the planet's orbital velocity.

For hyperbolic encounters with an object of mass M at an impact parameter b and relative velocity v_∞, the scattering angle θ can be determined from $\sin(\theta/2) = [1+b^2v_\infty^4/(GM)^2]^{-1/2}$. If the pre- and post-encounter velocities of the particle in the inertial frame are v_1 and v_2, respectively, and $\theta \ll 1$, show that

$$(v_2^2 - v_1^2) \approx 4(M_{\rm p}/M_\star)^2\epsilon_1^{-2}v_{\rm p}^5 v_\infty^{-3}.$$

Hence show that the new orbit has a semimajor axis with a corresponding ϵ_2 that satisfies

$$\epsilon_2 - \epsilon_1 \approx 4(M_{\rm p}/M_\star)^2\epsilon_1^{-2}(1-\beta)^{-1}(v_{\rm p}/v_\infty)^3.$$

Determine the longitude of the next conjunction between the particle and planet, both that which would have occurred in the absence of the encounter $\lambda_{\rm c1}$, and that including the perturbation of the encounter $\lambda_{\rm c2}$, in terms of β, ϵ_1 and ϵ_2.

The particle's orbit is unstable if $|\lambda_{\rm c1} - \lambda_{\rm c2}| > 2\pi$. Give a qualitative explanation for why this should be the case, and show that this means the orbits of particles with $\beta = 0$ near $j+1 : j$ resonances are unstable for

$$j > (2/729)^{1/7}(M_{\rm p}/M_\star)^{-2/7}.$$

Using the results obtained so far, sketch how you would expect the critical j above which $j+1 : j$ resonances are unstable to depend on β for a few different planet-to-star mass ratios, giving a physical explanation for any dependences.

3

A planet of mass M_1 and radius R_1 orbits a star of mass $M_\star \gg M_1$ and radius $R_\star \gg R_1$ on a nearly circular orbit with a semimajor axis a_1 and eccentricity $e_1 \ll 1$. The planet is surrounded by a spherical cloud of debris that extends to a distance of $R_1 = \eta R_{\rm h1} \gg R_\star$ from the planet, where $R_{\rm h1}$ is the planet's Hill radius. The planet's orbit is close to edge-on to our line-of-sight so that the planet and/or debris periodically transits in front of the star, with period P_1, causing the star's light-curve to exhibit dimming events each of duration D_1. Determine the constraint on P_1/D_1^3 in terms of the mean density of the star $\rho_\star$ for which it is the debris rather than the planet which is likely to be causing the dimming.

Show that if the debris causes the dimming then

$$M_1/M_\star = 3\left(\frac{\pi D_1}{\eta P_1}\right)^3.$$

An additional planet of mass $M_2 \gg M_1$ orbits the star in the same plane as M_1 on a circular orbit with semimajor axis a_2 and orbital period $P_2 = P_1\left(\frac{j+1}{j}\right)(1+\Delta)$, where $|\Delta|$ is large enough for the planets not to be in mean motion resonance, but small enough for perturbations due to the $j+1:j$ resonance to dominate the orbital evolution of planet 1. The relevant term in the disturbing function is $\mathcal{R}_1 = (GM_2/a_2)f(\alpha)e_1\cos\phi_1$, where f is a function of $\alpha = a_1/a_2$, $\phi_1 = (j+1)\lambda_2 - j\lambda_1 - \varpi_1$, λ_i is the mean longitude of planet i, and ϖ_1 is the longitude of pericentre of planet 1. Lagrange's planetary equations show that to lowest order

$$\dot{a}_1 = \frac{2}{n_1 a_1}(\partial\mathcal{R}_1/\partial\lambda_1), \qquad \dot{\epsilon}_1 = \frac{1}{2n_1 a_1^2}[-4a_1(\partial\mathcal{R}_1/\partial a_1) + e_1(\partial\mathcal{R}_1/\partial e_1)],$$

where n_1 is the mean motion of planet 1 and ϵ_1 is its mean longitude of epoch defined such that $\lambda_1 = n_1 t + \epsilon_1$. Show that

$$\dot{\lambda}_1 = n_1[1 + (M_2/M_\star)g(\alpha)e_1\cos\phi_1],$$

where g is a function of α that should be determined.

Solve for the evolution of n_1 and λ_1, and hence show that, if started at transit at $\phi_1 = 0$, the k-th transit of planet 1 in front of the star occurs after a time

$$P_1 k + A_{\rm t}\sin(2\pi t/P_{\rm t}),$$

where $A_{\rm t}/P_{\rm t} \approx \frac{3}{2\pi}\left(\frac{M_2}{M_\star}\right)\alpha f(\alpha)e_1/\Delta$.

Describe how these calculations would have changed if planet 1 had been in (rather than just near) the resonance, and comment on how this would affect the period and amplitude of the expected transit timing variations.

4

Consider a planetesimal belt in which the size distribution is defined with narrow logarithmically-spaced size bins, so that the total mass in bin k is $m_{\rm k}$ and all objects in the bin have diameter $\sim D_{\rm k}$. You may assume that the size distribution evolves due to catastrophic collisions, i.e., $\dot{m}_{\rm k} = \dot{m}_{\rm k}^{+c} - \dot{m}_{\rm k}^{-c}$, where $\dot{m}_{\rm k}^{+c}$ is the rate at which mass is gained in the k-th bin from catastrophic collisions in other bins and $\dot{m}_{\rm k}^{-c}$ is the rate at which mass is lost from the k-th bin due to catastrophic collisions. The size distribution of fragments created in catastrophic collisions is scale-independent. Show that when the size distribution has reached steady state the mass loss rate from the bins is independent of size.

If the size distribution is a power law defined by a slope α such that $n(D) \propto D^{-\alpha}$, where $n(D)dD$ is the number of planetesimals in the size range D to $D + dD$, and $M_{\rm tot}$ is the total mass in the distribution, then the rate of collisions onto objects of size $D_{\rm k}$ with specific incident energy greater than Q is $R(D_{\rm k}, Q) \propto M_{\rm tot} Q^{(1-\alpha)/3} D_{\rm k}^{3-\alpha}$. If the dispersal threshold $Q_{\rm D}^\star \propto D^b$, show that the steady state size distribution has a slope $\alpha = \frac{21+b}{6+b}$.

Show that the timescale for the size distribution in bin k to reach steady state is $\sim 1/R(D_{\rm k}, Q_{\rm D}^\star)$.

The planetesimal belt starts with a power law size distribution defined by $\alpha = 7/2$ between sizes $D_{\rm min}$ and $D_{\rm max}$. The dispersal threshold is $Q_{\rm D}^\star = Q_a D^{-a} + Q_b D^b$, where $a = 1/2$ and $b = 3/2$ are the slopes in the strength and gravity regimes respectively, which has a minimum at $D_{\rm w}$ where $D_{\rm min} < D_{\rm w} < D_{\rm max}$. Sketch the catastrophic collision timescale as a function of $D_{\rm k}$ for this primordial distribution, and hence describe how the size distribution evolves by sketching the distribution at a few representative epochs that should be noted on the timescale plot.

After some time T, when the transition between the steady state and primordial parts of the size distribution is at $D_{\rm t}$ where $D_{\rm w} < D_{\rm t} < D_{\rm max}$, the belt undergoes a dynamical depletion in which mass in all bins is reduced by a factor $f \gg 1$. Describe the subsequent evolution of the size distribution due to collisions.

The size distribution is measured shortly after the depletion in the absence of knowledge that the depletion occurred. Determine how long it would be inferred that the system had been undergoing collisional evolution, and comment on whether the depletion factor f can be determined from the size distribution.

Show that the number of *rubblising* collisions, defined as those with sufficient energy to cause catastrophic disruption in the absence of self-gravity, that objects of size $D_{\rm t}$ would have undergone at time T is $(D_{\rm t}/D_{\rm w})^{4/3}$.

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