UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 9:00 am to 12:00 pm

PAPER 62

GALACTIC ASTRONOMY AND DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The collisionless Boltzmann equation has the form

$$v_i \frac{\partial F}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial F}{\partial v_i} = 0,$$

where F is the distribution function and Φ is the gravitational potential. Using this, derive *the Jeans equations* in the form

$$\frac{\partial}{\partial x_i}\rho\langle v_i v_j\rangle = -\rho\frac{\partial \Phi}{\partial x_j},$$

where ρ is the stellar density and angled brackets denote averages over the distribution function.

Hence, derive *the tensor virial theorem*

$$2K_{ij} + W_{ij} = 0,$$

where the kinetic energy K_{ij} and potential energy W_{ij} tensors are

$$\begin{aligned} K_{ij} &= \frac{1}{2}\int \rho\langle v_i v_j\rangle dV, \\ W_{ij} &= -\int \rho x_i \frac{\partial \Phi}{\partial x_j} dV. \end{aligned}$$

By denoting the traces as $K = K_{ii}$ and $W = W_{ii}$, show that the total energy E of the system satifies

$$E = -K = \frac{W}{2}.$$

Now consider a system with initial total mass $M_{\rm I}$, total energy $E_{\rm I}$, mean square velocity of stars $\langle v_{\rm I}^2\rangle$ and gravitational radius $R_{\rm I}$. Show that

$$E_{\rm I} = -\frac{1}{2}M_{\rm I}\langle v_{\rm I}^2\rangle = -\frac{GM_{\rm I}^2}{2R_{\rm I}}.$$

Suppose that systems are accreted with energies totally $E_{\rm A}$, masses totalling $M_{\rm A}$, and mean square speeds averaging $\langle v_{\rm A}^2\rangle$. If we define the fractions $\eta = M_{\rm A}/M_{\rm I}$ and $\epsilon = \langle v_{\rm A}^2\rangle/\langle v_{\rm I}^2\rangle$, show that the final energy of the system is

$$E_{\rm F} = -\frac{1}{2}M_{\rm I}\langle v_{\rm I}^2\rangle(1+\epsilon\eta),$$

explaining carefully any assumptions made.

Show that the ratio of final to initial mean square speeds is

$$\frac{\langle v_{\rm F}^2\rangle}{\langle v_I^2\rangle} = \frac{1+\eta\epsilon}{1+\eta}.$$

Show further that the ratio of final to initial gravitational radii is

$$\frac{R_{\rm F}}{R_{\rm I}} = \frac{(1+\eta)^2}{1+\eta\epsilon}.$$

If the total mass of the system increases by a factor of 2, show that one equal mass merger causes the radius to increase by a factor of 2, but many minor mergers cause it to increase by a factor of 4.

2

Explain what is meant by *the phase space distribution function* of a galaxy.

A galaxy model has potential

$$\Phi(r) = v_0^2 \log\left[\frac{a+r}{a}\right],$$

where v_0 and a are constants. Show that the density of the model is

$$\rho(r) = \frac{v_0^2}{4\pi G}\frac{2a+r}{r(a+r)^2}.$$

Derive and sketch the rotation curve of the model. Is it realistic?

Demonstrate that stars moving in the galaxy's potential conserve their energy per unit mass E and their angular momentum per unit mass $\underline{L}$.

Prove *Jeans Theorem*, namely that the phase space distribution function of a steady-state galaxy depends only on the integrals of motion.

Show that a possible phase space distribution function for the galaxy model is

$$F(E,L) = \frac{1}{8\pi^3 GaL}\left[\exp(-E/v_0^2) + 2\exp(-2E/v_0^2)\right],$$

where $L = |\underline{L}|$.

Show that the second velocity moments or velocity dispersions are

$$\begin{aligned}\langle v_r^2\rangle &= \frac{v_0^2}{2}\frac{3a+2r}{2a+r},\\ \langle v_\theta^2\rangle &= \langle v_\phi^2\rangle = \frac{v_0^2}{4}\frac{3a+2r}{2a+r}.\end{aligned}$$

Show also that the mixed moments all vanish, that is

$$\langle v_r v_\theta\rangle = \langle v_r v_\phi\rangle = \langle v_\theta v_\phi\rangle = 0.$$

Interpret your results in terms of the orbital structure of the model.

Hint: You are reminded of the standard integral ($\alpha > 0$)

$$\int_{-\infty}^{\infty} \exp(-\alpha v^2)dv = \sqrt{\frac{\pi}{\alpha}}$$

3

A star moves on a circular orbit in a galaxy with the axisymmetric potential $\Phi(R, z)$. If L_z is the star's angular momentum component parallel to the symmetry axis, show that the radius R_c of the circular orbit in the equatorial plane is given by

$$\left.\frac{\partial \Phi}{\partial R}\right|_{(R_\mathrm{c},0)} = \frac{L_z^2}{{R_\mathrm{c}}^3}$$

Now suppose the star's motion is perturbed about the circular orbit. Letting $x = R - R_\mathrm{c}$ and letting z denote denote displacement perpendicular to the equatorial plane, derive the equations of the *epicyclic approximation*

$$\ddot{x} = -\kappa^2 x, \qquad \ddot{z} = -\nu^2 z,$$

where a superscript dot means a derivative with respect to time, whilst κ and ν are the epicyclic and vertical frequencies respectively. Explain carefully any assumptions made.

If the circular frequency is Ω, show that

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2.$$

Explain why $\Omega \lesssim \kappa \lesssim 2\Omega$ for applications in galactic dynamics.

By erecting Cartesian coordinates (x, y) at the guiding center, show that

$$x = X\cos(\kappa t + \chi), \qquad y = -Y\sin(\kappa t + \chi),$$

where χ is a constant phase, and the amplitudes X and Y satisfy

$$\frac{X}{Y} = \frac{\kappa}{2\Omega}.$$

Interpret your result geometrically.

The Oort's constants in the solar neighbourhood are defined by

$$\begin{aligned} A &= -\frac{1}{2}R\frac{d\Omega}{dR} = 14.5\ \mathrm{kms^{-1}kpc^{-1}}, \\ B &= -\left(\frac{1}{2}R\frac{d\Omega}{dR} + \Omega\right) = -12\ \mathrm{kms^{-1}kpc^{-1}}. \end{aligned}$$

Use these to estimate X/Y for the Sun's orbit.

Let $\Delta\psi$ denote the increment in azimuthal angle for one complete radial oscillation. Show that, in the epicyclic approximation,

$$\Delta\psi = 2\pi\left(4 + \frac{d\log\Omega^2}{d\log R}\right)^{-1/2}.$$

Estimate $\Delta\psi$ for the Sun's orbit.

4

A spherical galaxy has density $\rho(r)$, where r is spherical polar radius. Show that the gravitational potential is

$$\Phi(r) = -4\pi G \left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r'dr'\right].$$

Show that the circular speed is

$$v_{\rm c}^2 = \frac{GM(r)}{r},$$

where $M(r)$ is the mass enclosed within r.

Now consider an infinitesimally thin axisymmetric disk with surface density $\Sigma(R)$, where R is cylindrical polar radius. Show that the gravitational potential in the disk is

$$\Phi(R) = -G\int_0^\infty \Sigma(R')R'dR' \int_0^{2\pi} \frac{d\phi'}{|\underline{x} - \underline{x}'|},$$

where $\underline{x}' = (R', \phi')$ and $\underline{x} = (R, \phi)$ in polar coordinates.

By expanding in Legendre polynomials, show that the circular speed in the disk is

$$\begin{aligned} v_{\rm c}^2 \quad = \quad & \frac{GM(R)}{R} + 2G\sum_{k=1}^{\infty} \alpha_{2k} \left[(2k+1)R^{-(2k+1)} \int_0^R \Sigma(R')R'^{2k+1} dR' \right. \\ & \left. - 2kR^{2k} \int_R^\infty \Sigma(R')R'^{-2k} dR'\right], \end{aligned}$$

where

$$\alpha_n = \pi\left(\frac{(n)!}{2^n((n/2)!)^2}\right)^2, \qquad (1)$$

and $M(R)$ is the mass enclosed within radius R.

In many galaxies, the disk surface density Σ falls off exponentially with R. Explain why the rotation curve of such an exponential disk approaches the Keplerian limit from above.

Find an example of an infinitesimally thin disk for which

$$v_{\rm c}^2 = \frac{GM(R)}{R},$$

and plot its rotation curve.

Hint: You may find useful the following expansion

$$\frac{1}{|\underline{x} - \underline{x}'|} = \sum_{k=0}^{\infty} \frac{R_<^k}{R_>^{k+1}} P_k(\cos\gamma),$$

where $R_< = \min(|\underline{x}|, |\underline{x}'|)$ *and* $R_> = \max(|\underline{x}|, |\underline{x}'|)$ *and* γ *is the angle between* $\underline{x}$ *and* $\underline{x}'$.

You may also assume without proof the following integrals of Legendre polynomials

$$\int_{-\pi}^{\pi} P_n(\cos\phi)d\phi = \begin{cases} 0, & \text{if } n \text{ odd,} \\ 2\alpha_n, & \text{if } n \text{ even,} \end{cases}$$

where α_n *is given in eq. (1).*

END OF PAPER