UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 9:00 am to 11:00 am

PAPER 60

MAGNETOHYDRODYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.
Extra credit will be given for complete answers.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Consider a straight flux tube in a perfect gas. In cylindrical polar coordinates (s, ϕ, z) the field takes the form $\mathbf{B} = (0, 0, B_z(s))$. If gravity is ignored and the tube is in equilibrium with the pressure gradient find an expression for the (non-negative) pressure $p(s)$. Show that if $B_z \equiv 0$, $p = p_e$ for $s \geqslant s_0$ then $|B_z(s)| \leqslant B_p \equiv \sqrt{2\mu_0 p_e}$.

(ii) Now suppose we have a more general equilibrium in which $\mathbf{B} = (0, B_\phi(s), B_z(s))$ with $|\mathbf{B}| \equiv 0$ for $s \geqslant s_0$ as before. It may be assumed that $B_\phi(0) = 0$. Show that $B_z(0)^2 = B_p^2 - 2\mu_0 p(0) + \int_0^{s_0} \frac{2}{s} B_\phi^2(s)\, ds$, so that $|B_z(0)|$ can be greater than B_p, but that the mean square field is bounded in the sense that

$$\frac{2}{s_0^2} \int_0^{s_0} s B_z(s)^2 \, ds < B_p^2.$$

(iii) In the special case where $B_\phi = \gamma s B_z / s_0$ show also that

$$B_z^2(0) \leqslant (\gamma^2 + 1) B_p^2.$$

2

Small amplitude MHD waves with solenoidal velocity $\mathbf{u}$ and perturbation field $\mathbf{b}$ in a diffusionless rotating fluid with angular velocity $\Omega\hat{\mathbf{z}}$ in the presence of a uniform current $\mathbf{J} = J\hat{\mathbf{z}}$ and associated magnetic field $\mathbf{B} = \frac{1}{2}Js\hat{\boldsymbol{\phi}}$ obey the equations

$$\frac{\partial \mathbf{u}}{\partial t} + 2\Omega\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \frac{1}{\mu_0 \rho}(\mathbf{B}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{B}),$$

$$\frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B}.$$

Consider solutions of the form $\mathbf{u}(s,\phi,z,t) = \tilde{\mathbf{u}}(s,z)e^{\sigma t + \mathrm{i}m\phi}$, etc., where m is an integer. Then verify that

$$\mathbf{B}\cdot\nabla\mathbf{b} + \mathbf{b}\cdot\nabla\mathbf{B} = J\hat{\mathbf{z}} \times \mathbf{b} + \frac{1}{2}\mathrm{i}mJ\mathbf{b},$$

and find the corresponding expression for $\mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B}$. By eliminating $\mathbf{b}$, show that $\mathbf{u}$ obeys the equation

$$\lambda\mathbf{u} + \nu\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p,$$

where

$$\lambda = \sigma + \frac{m^2 J^2}{4\mu_0\rho\sigma}, \quad \nu = 2\Omega - \frac{\mathrm{i}mJ^2}{2\mu_0\rho\sigma}.$$

It may be shown that when boundary conditions are taken into account, solutions of this equation can only exist if $\lambda = \mathrm{i}\delta\nu$ with δ real and $|\delta| < 1$. Find an expression for σ in terms of δ and the other parameters, and hence show that σ is purely imaginary (corresponding to stable waves), unless $m = 1$ and J is sufficiently large.

3

Consider a mean field dynamo model with anisotropic α-effect, so that

$$\alpha_{ij} = \alpha_0\delta_{ij} - (\alpha_0 - \alpha_1)\delta_{i3}\delta_{j3},$$

where $\alpha_{0,1}$ are constants, in Cartesian coordinates (x, y, z).

Seek steady solutions of the form $\mathbf{B} \propto e^{ikx+ily+imz}$. Find a relation between $\gamma = \alpha_0/\eta$, $\delta = \alpha_1/\alpha_0$, $q^2 = k^2 + l^2$ and m^2 for such solutions to be possible.

Now assume that q^2 is fixed by the boundary conditions in x, y but that m is freely varying. Find the minimum value of γ^2 as a function of m^2, and the associated value of m^2, in the range $0 < \delta < 1$.

4

State carefully Backus' necessary condition for dynamo action. Indicate how to adapt the proof to show that the growth rate for any dynamo is no greater than the maximum strain rate.

Now consider a simple one-dimensional $\alpha\Omega$ dynamo model

$$\dot{A} = \alpha_0 f(t) B - \eta k^2 A, \qquad \dot{B} = ik\Omega A - \eta k^2 B,$$

where Ω gives the strain rate of the shear and α_0 is due to the small scale velocity field. $f(t)$ is bounded in time and $|f(t)| \leqslant 1$. k is a variable parameter with k^{-1} giving the length scale of the growing field.

Show by forming equations for the time derivatives of $|A|^2$ and $|B|^2$ that the maximum possible rate of growth of this model dynamo scales as $\Omega^{2/3}$ as $\Omega \to \infty$. Why is this much smaller than the bound $\sim \Omega$ suggested by the Backus criterion?

END OF PAPER