UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2015 9:00 am to 12:00 pm

PAPER 6

FUNCTIONAL ANALYSIS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Define the (real) normed spaces $L_p(\Omega, \mathcal{F}, \mu)$ for $1 \leqslant p \leqslant \infty$, and show that they are complete.

State a theorem identifying the dual space of $L_p(\Omega, \mathcal{F}, \mu)$ in the case $1 \leqslant p < \infty$. Prove this theorem in the case when $1 < p < \infty$ and when $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. [Results from measure theory can be used without proof.]

Prove carefully that for $1 < p < \infty$ the Banach space $L_p(\Omega, \mathcal{F}, \mu)$ is reflexive. [*Hint: It is not enough to show that* $L_p(\Omega, \mathcal{F}, \mu)$ *is isometrically isomorphic to its second dual.*]

2

Define the weak topology of a normed space. State and prove Mazur's theorem. [Any version of the Hahn–Banach theorem may be used without proof.]

Define the weak-star topology of a dual space. State and prove the Banach–Alaoglu theorem. [Results from general topology may be used without proof.]

What is the subspace topology on X induced by the weak-star topology on X^{**}? [Here we identify X with its image in X^{**} under the canonical embedding.]

Let C be a bounded, convex subset of a Banach space X. Show that $\overline{C}$, the norm-closure of C in X, is weakly compact if and only if $\overline{C}^{w^*}$, the weak-star-closure of C in X^{**}, is contained in X.

Let $T\colon X \to Y$ be a bounded linear map between Banach spaces. Show that $T^*\colon Y^* \to X^*$ is continuous with respect to the weak-star topologies of Y^* and X^*. Show that $\overline{T(B_X)}$ is weakly compact if and only if $T^{**}(X^{**}) \subset Y$. [*Hint for the 'only if' part: Goldstine's theorem.*]

3

[In this question all vector spaces are over the field of complex numbers.]

What is an *extreme point* of a convex set in a vector space?

State and prove the Krein–Milman theorem for compact, convex sets in locally convex spaces. [Any version of the Hahn–Banach theorem may be used without proof.]

Describe without proof the set of extreme points of the closed unit ball of $C(K)^*$ for a compact Hausdorff space K. State and prove the Banach–Stone theorem.

Is there a Banach space whose dual space is isometrically isomorphic to c_0 or to $L_1[0,1]$? Are the spaces $C[0,1]$ and $C([0,1] \cup [2,3])$ isometrically isomorphic? Justify your answers.

4

Develop the theory of commutative, unital Banach algebras up to and including the Gelfand Representation Theorem. Calculate the Gelfand map for the algebra of continuous functions on a compact Hausdorff space. [Basic properties of Banach algebras and elementary spectral theory, including the Gelfand–Mazur theorem on normed division algebras, can be assumed without proof.]

What can you say about the Gelfand map for a commutative, unital C*-algebra?

5

Let X be a real or complex vector space and p be a semi-norm on X. Let Y be a subspace of X and g be a linear functional on Y such that $|g(y)| \leqslant p(y)$ for all $y \in Y$. Prove that there exists a linear functional f on X such that $f|_Y = g$ and $|f(x)| \leqslant p(x)$ for all $x \in X$. [No version of the Hahn–Banach theorem can be used without proof.]

Let Y be a subspace of a normed space X. Let $z \in X \setminus Y$ with $d = d(z, Y) = \inf\{\|z - y\| : y \in Y\} > 0$. Show that there exists $f \in X^*$ such that $Y \subset \ker f$, $\|f\| = 1$ and $f(z) = d$.

Prove the existence of $L \in \ell_\infty^*$ with the following properties. [Here ℓ_∞ is the *real* space of bounded sequences, and c denotes the subspace of convergent sequences.]

(a) $\|L\| = 1$.

(b) If $x = (x_n) \in \mathrm{c}$, then $L(x) = \lim_{n\to\infty} x_n$.

(c) If $x = (x_n) \in \ell_\infty$ and $x' = (x_2, x_3, x_4, \dots)$, then $L(x) = L(x')$.

[*Hint: Consider* $Y = \{x - x' : x \in \ell_\infty\}$ *and* $z = (1, 1, 1, \dots)$.]

END OF PAPER