UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 1:30 pm to 4:30 pm

PAPER 59

EXTRASOLAR PLANETS — ATMOSPHERES AND INTERIORS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Spectroscopy of Transiting Exoplanets

a. Starting from the radiative transfer equation, show that the specific intensity of emergent radiation at the top of a plane parallel atmosphere in local thermodynamic equilibrium (LTE) is of the form

$$I_\lambda = I_\lambda(0)e^{(\tau_2-\tau_1)/\mu} + e^{\tau_2/\mu}\int_{\tau_2}^{\tau_1} \frac{B_\lambda e^{-\tau/\mu}}{\mu} d\tau, \tag{1}$$

where, τ_1 and τ_2 are the optical depths at the bottom and top of the atmosphere respectively ($\tau_1 \geqslant \tau_2$, as viewed from the top), μ is the direction cosine relative to the normal to the surface, and $I_\lambda(0)$ is the specific intensity at the bottom of the atmosphere, and B_λ is the Planck function.

Show that the emission spectrum of an isothermal semi-infinite atmosphere (i.e. $\tau_1 \to \infty$) is a blackbody spectrum.

b. Derive the expression for the eclipse depth, relative to out-of-eclipse flux, as a function of wavelength for a transiting exoplanet observed during secondary eclipse as the planet disappears behind the host star. Assume the planet and star to be blackbodies with temperatures T_p and T_s, and to have radii R_p and R_s, respectively. [*Hint: The Planck function is given by*

$$B_\lambda = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda k_b T} - 1}, \tag{2}$$

where the symbols have their standard meanings.]

Based on the above result, sketch a qualitative plot of the planet-star flux ratio as a function of wavelength for a typical hot Jupiter. Explain what happens in the long wavelength (Rayleigh-Jeans) limit.

c. A transmission spectrum was observed for an exoplanet orbiting a sun-like star in the wavelength range of 1.2 - 1.7 μm which contains a strong H_2O band. The continuum level of the absorption was found to be at 1% and the amplitude of the H_2O absorption peak was 0.03%.
(i) What is the approximate size of the planet relative to that of Earth or Jupiter? What is the most dominant molecule expected in its atmosphere?
(ii) Assuming the gravity of the planet is similar to that of its nearest solar system analogue, what is the approximate temperature of the planetary atmosphere in the region the spectrum is probing? You may express your answer in terms of the atmospheric scale height (H_0) of a solar system planet. Make and state any assumptions required.

2 Energy Transport and Temperature Profiles

a. The temperature gradient (dT/dz) of a plane-parallel atmosphere in radiative equilibrium and heated only from an interior energy source is given by the diffusion approximation in the large optical depth limit,

$$\frac{dT}{dz} = -\frac{3\bar{\kappa}\rho}{16\sigma T^3}F, \tag{1}$$

where, $\bar{\kappa}$ is the mean opacity, ρ is the density, σ is the Stefan-Boltzmann constant, and F is the internal flux. Use this result to derive the temperature as a function of the optical depth and an internal effective temperature. Show that for a weakly irradiated planetary atmosphere the temperature decreases monotonically with altitude and approaches an isotherm for low optical depth. Which type of exoplanets known exhibit such behavior, and how are they detected? How does such a temperature profile differ from that of a strongly irradiated hot Jupiter? Provide an illustrative sketch along with your answer.

b. A bubble of hot ideal gas expands as it rises adiabatically in a planetary atmosphere in hydrostatic equilibrium. Show that the temperature of the gas inside the bubble (T_g) varies as $dT_g/dz = -g/C_p$, where g is the gravity of the planet and C_p is the specific heat capacity at constant pressure. You may note that for an adiabatic expansion the pressure and volume are related as PV^γ = constant, where $\gamma = C_p/C_v$ is the ratio of specific heats.

   Explain why the condition for the temperature profile of an atmosphere to be unstable against convection is given by $dT/dz \leqslant -g/C_p$. Which kinds of planets are expected to have such temperature profiles and in which parts of their atmospheres or interiors?

c. The thermal spectrum of the dayside of a hot Jupiter revealed a spectrum with molecular emission features as peaks over a blackbody continuum. What is responsible for causing the emission peaks in the spectrum besides the chemical composition? What is responsible for the continuum? Show that in the region of the atmosphere where the emission peaks originate the mechanism of vertical energy transport is radiative, i.e. not convective.

d. What is a thermal inversion in a planetary atmosphere, and what causes it? Which solar system planets have thermal inversions? List the factors that influence thermal inversions in the atmospheres of hot Jupiters.

3 Planetary Interiors

a. When studying planetary internal structure it is often useful to consider polytropic models which allow semi-analytic solutions. For a polytropic model, the pressure and density are related by $P = K\rho^{1+1/n}$, where K and n are constants; n is called the polytropic index. For polytropes, the equations of mass conservation and hydrostatic equilibrium can be combined to derive the Lane-Emden equation which is given by

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n. \tag{1}$$

In this formalism, the radial distance coordinate and density are defined as $r = \alpha\xi$ and $\rho = \rho_c\theta^n$, respectively, $\alpha = C_1\rho_c^{(1-n)/2n}$, where C_1 is a constant, and ρ_c is the central density. Use the above information and the mass conservation equation to show that the total mass (M) and radius (R) of a planetary object are related by

$$M^{(n-1)}R^{(3-n)} = C_2 \tag{2}$$

where C_2 is a constant independent of ρ_c.

What physical situations in planetary interiors can be described by polytropic solutions with indices $n = 0$ and $n = 3/2$? Give an example of an astronomical object to which each index applies.

b. Draw a rough sketch of a theoretical $M - R$ curve for astronomical objects ranging in mass from ice giants to low-mass stars, showing the approximate power-law exponents where applicable. Identify the transitions between different object types and explain the reasons behind the transitions. Briefly comment on the source(s) of energy and the mechanism of energy transport in the interiors of each class of objects.

c. Consider a spherical rocky planet with a density that is constant throughout the planet's interior. Show that the pressure in the interior is given by

$$P(r) = P_c(1 - r^2/R_p^2), \tag{3}$$

where, r is the radial coordinate in the interior, R_p is the bulk radius, and P_c is the central pressure. Derive P_c as a function of the surface gravity of the planet and fundamental constants.

4 Short Essay Questions:
Candidates attempting this question should attempt all parts (a-j).

a. What are the three most abundant molecules (besides H_2 and He) expected in hot Jupiter atmospheres? How do the equilibrium abundances of those molecules vary as a function of temperature, metallicity, and C/O ratio for a pressure of 1 bar?

b. Describe in 1-3 sentences each three non-equilibrium chemical processes that can occur in exoplanetary atmospheres and state one example for each process.

c. State three observable signatures of clouds or hazes in exoplanetary atmospheres. State the type of observation and the approximate wavelength range in which each signature can be observed.

d. Derive the expression for the equilibrium temperature of a planet as a function of its system parameters. Make and state any assumptions required.

e. What is a habitable zone? State four factors that influence the habitability of terrestrial planets. What is the extent of the habitable zone in the solar system?

f. What are the characteristics of an ideal biosignature gas? What are primary and secondary metabolic byproducts? What are the two unique biosignature gases in the Earth's atmosphere?

g. What is the dominant mode of energy transport in each of these regions: (a) interiors of gas giants, (b) interiors of rocky planets, (c) atmospheres of giant planets between 0.1-10 bar, (d) atmospheres of hot Jupiters between 0.1-10 bar? What are the typical atmospheric temperatures of Earth, Jupiter, hot Jupiter, Sun?

h. Briefly describe what is plate tectonics, what causes it, and what are its effects? State three factors that influence the possibility of plate tectonics on super-Earths.

i. Name three exoplanet detection methods. Which detection methods are also useful for observing the atmospheres of exoplanets? State two factors that make it possible to detect exoplanets with large orbital separations?

j. What is the "Inflated hot Jupiters" problem? What are the two broad classes of solutions that have been considered to solve the problem? List 2 specific solutions in each class and their limitations, if any.

END OF PAPER