UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 9:00 am to 12:00 pm

PAPER 57

ASTROPHYSICAL FLUID DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

You are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u} \,,$$
$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u} \,,$$
$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} p + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \,,$$
$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \,,$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \,,$$
$$\nabla^2 \Phi = 4\pi G \rho \,.$$

1

(a) Formulate the equations governing the steady, spherically symmetric, adiabatic flow of an unmagnetized, non-self-gravitating perfect gas in a gravitational potential Φ that depends only on the spherical radius r.

(b) If $\Phi = -Ar^{-\beta}$, where A and β are positive constants, show that a necessary condition for either (i) an inflow that starts from rest at $r = \infty$ or (ii) an outflow that reaches $r = \infty$ to pass through a sonic point is

$$\gamma < f(\beta) \,,$$

where $\gamma > 1$ is the adiabatic exponent and $f(\beta)$ is a function to be determined.

(c) Assuming that this condition is satisfied, calculate the accretion rate of a transonic accretion flow in terms of A, β, γ and the density and sound speed at $r = \infty$. Evaluate your expression in each of the limits $\gamma \to 1$ and $\gamma \to f(\beta)$.

[You may find it helpful to define $\delta = \gamma - 1$. You may assume that

$$\lim_{\epsilon \to 0} (1 - \epsilon x)^{-1/\epsilon} = \mathrm{e}^x \qquad \text{and} \qquad \lim_{\epsilon \to 0} (\epsilon x)^{-\epsilon} = 1 \quad \text{for } x \neq 0 \,.]$$

2

In this question all fields may be assumed to be independent of y in Cartesian coordinates (x, y, z).

(a) A twisted magnetic flux tube in a plane-parallel atmosphere has a magnetic field that is independent of y. Explain why this can be written as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (\psi\, \boldsymbol{e}_y) + B_y\, \boldsymbol{e}_y$$

in terms of a magnetic flux function ψ. Show that the Lorentz force per unit volume is

$$-\frac{1}{\mu_0}\left(\nabla^2\psi\, \boldsymbol{\nabla}\psi + B_y \boldsymbol{\nabla} B_y + \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla} B_y\right).$$

(b) If the tube is in magnetostatic equilibrium, show that $B_y = B_y(\psi)$ and

$$\frac{1}{\mu_0}\left(\nabla^2\psi + B_y \frac{\mathrm{d}B_y}{\mathrm{d}\psi}\right)\boldsymbol{\nabla}\psi + \boldsymbol{\nabla} p + \rho \boldsymbol{\nabla}\Phi = \boldsymbol{0}.$$

(c) Now suppose instead that the tube rises through the atmosphere and is accompanied by a velocity field $\boldsymbol{u}$. Show that ψ and B_y evolve according to

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}B_y}{\mathrm{D}t} = \boldsymbol{B} \cdot \boldsymbol{\nabla} u_y - B_y \boldsymbol{\nabla} \cdot \boldsymbol{u}.$$

Deduce that, if the initial conditions at $t = 0$ are such that $u_y = 0$ and $B_y = f(\psi)$ is a function of ψ only, then, provided that $\boldsymbol{\nabla} \cdot \boldsymbol{u} = g(\psi, t)$ is a function of ψ and t only, no force or motion in the y direction will result during the rising of the tube.

3

In Cartesian coordinates (x, y, z), a non-self-gravitating ideal MHD flow in the presence of a uniform horizontal shear flow has the form

$$\boldsymbol{u} = \boldsymbol{v}(z,t) + ax\,\boldsymbol{e}_y\,, \qquad \boldsymbol{B} = \boldsymbol{B}(z,t)\,, \qquad \rho = \rho(z,t)\,,$$

where a is a constant describing the background shear. The flow is isothermal, with $p = c_{\mathrm{s}}^2\rho$ and $c_{\mathrm{s}} =$ constant, and the gravitational field is of the form $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi = -g(z)\,\boldsymbol{e}_z$.

(a) Show that B_z is constant, and derive the equations

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)\rho = -\rho\frac{\partial v_z}{\partial z}\,,\\
&\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)v_x = \frac{B_z}{\mu_0\rho}\frac{\partial B_x}{\partial z}\,,\\
&\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)v_y + av_x = \frac{B_z}{\mu_0\rho}\frac{\partial B_y}{\partial z}\,,\\
&\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)v_z = -g - \frac{1}{\rho}\frac{\partial}{\partial z}\left(p + \frac{B^2}{2\mu_0}\right)\,,\\
&\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)B_x = B_z\frac{\partial v_x}{\partial z} - B_x\frac{\partial v_z}{\partial z}\,,\\
&\left(\frac{\partial}{\partial t} + v_z\frac{\partial}{\partial z}\right)B_y = B_z\frac{\partial v_y}{\partial z} + aB_x - B_y\frac{\partial v_z}{\partial z}\,.
\end{aligned}$$

(b) Deduce the associated total energy equation in the form

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial z} = S\,,$$

where

$$E = \rho\left[\tfrac{1}{2}v^2 + \Phi + c_{\mathrm{s}}^2\ln\left(\frac{\rho}{\rho_0}\right)\right] + \frac{B^2}{2\mu_0}$$

is the energy per unit length in the z direction (not including the energy of the background shear flow), ρ_0 is an arbitrary reference density, F is an energy flux in the z direction and S is a source term proportional to a. Give explicit expressions for F and S.

(c) In the case of a steady flow, take linear combinations of the equations to show that

$$\left[v_z^4 - \left(c_{\mathrm{s}}^2 + v_{\mathrm{a}}^2\right)v_z^2 + c_{\mathrm{s}}^2 v_{\mathrm{a}z}^2\right]\frac{1}{v_z}\frac{\mathrm{d}v_z}{\mathrm{d}z} = g\left(v_{\mathrm{a}z}^2 - v_z^2\right) + av_{\mathrm{a}y}\left(v_x v_{\mathrm{a}z} - v_z v_{\mathrm{a}x}\right),$$

where $\boldsymbol{v}_{\mathrm{a}}$ is the Alfvén velocity. Discuss the physical significance of the form of this equation if v_z represents the velocity of an outflow that accelerates from very low to very high velocities as z increases.

4

A hypothetical model of a star consists of a perfect gas sphere of mass M, radius R and uniform density ρ, in hydrostatic equilibrium under its own gravity, with no magnetic field, and surrounded by empty space. The adiabatic exponent of the gas is γ. For $r \leqslant R$, the inward gravitational acceleration is $g = \omega_\mathrm{d}^2 r$ and the pressure is $p = \frac{1}{2}\rho\omega_\mathrm{d}^2(R^2 - r^2)$, where $\omega_\mathrm{d} = (GM/R^3)^{1/2} = (4\pi G\rho/3)^{1/2}$ is the dynamical frequency of the star.

(a) Formulate, by any method, the linearized equations for small perturbations to the equilibrium state.

(b) Assume that the displacement and associated Eulerian perturbations have the form

$$\begin{aligned}
\boldsymbol{\xi} &= U(r)F\,\boldsymbol{r} + V(r)\boldsymbol{\nabla}F\,, \\
\delta\rho &= \hat{\rho}(r)F\,, \\
\delta p &= \hat{p}(r)F\,, \\
\delta\Phi &= \hat{\Phi}(r)F\,,
\end{aligned}$$

where (r, θ, ϕ) are spherical polar coordinates, $\boldsymbol{r} = r\,\boldsymbol{e}_r$ is the position vector and

$$F = r^l Y_l^m(\theta, \phi)\,\mathrm{e}^{-\mathrm{i}\omega t}\,,$$

where $l \geqslant 0$ is an integer and Y_l^m is a spherical harmonic function such that $\nabla^2 F = 0$. Show that the various functions of r satisfy the ordinary differential equations

$$\begin{aligned}
\rho\omega^2 U r &= \hat{\rho}\omega_\mathrm{d}^2 r + \rho\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}r} + \frac{\mathrm{d}\hat{p}}{\mathrm{d}r}\,, \\
\rho\omega^2 V &= \rho\hat{\Phi} + \hat{p}\,, \\
\hat{\rho} &= -\rho\Delta\,, \\
\hat{p} &= \rho\omega_\mathrm{d}^2(Ur^2 + lV) - \gamma p\Delta\,, \\
\frac{\mathrm{d}^2\hat{\Phi}}{\mathrm{d}r^2} &+ \frac{2(l+1)}{r}\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}r} = 4\pi G\hat{\rho}\,,
\end{aligned}$$

in $r < R$, where

$$\Delta = r\frac{\mathrm{d}U}{\mathrm{d}r} + (l+3)U + \frac{l}{r}\frac{\mathrm{d}V}{\mathrm{d}r}\,.$$

(c) Assume further that there exist solutions of these equations in which V, $\hat{p}$ and $\hat{\Phi}$ are even polynomials of degree n in r, while U, $\hat{\rho}$ and Δ are even polynomials of degree $n-2$ in r, where $n \geqslant 2$ is an even integer. By examining the highest power of r in each of the above equations, deduce that

$$\omega^4 + \left[4 - \tfrac{1}{2}\gamma n(2l + n + 1)\right]\omega_\mathrm{d}^2\omega^2 - l(l+1)\omega_\mathrm{d}^4 = 0\,.$$

Use this relation to analyse the stability of the model to perturbations of this type, and comment on the results.

[*Hint: In manipulating the algebraic equations it may be found easiest to eliminate first the coefficients of the highest powers of* r *in* U *and* V.]

[You need not consider the boundary conditions. The square of the (Brunt–Väisälä) buoyancy frequency is

$$N^2 = g\left(\frac{1}{\gamma}\frac{\mathrm{d}\ln p}{\mathrm{d}r} - \frac{\mathrm{d}\ln\rho}{\mathrm{d}r}\right) .]$$

END OF PAPER