UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2015 9:00 am to 11:00 am

PAPER 56

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Show, by constructing an atlas of charts, that the n–dimensional ellipsoid

$$\mathcal{E}^n = \{\mathbf{r} \in \mathbb{R}^{n+1}, \sum_{i=1}^{n+1} {a_i}^2 {r_i}^2 = 1\}, \quad a_i \neq 0$$

is a smooth manifold.

Consider the left–translations on the Lie group $SU(2)$ to show that S^3 admits three independent non–vanishing global vector fields.

2

Define a Maurer–Cartan one–form ρ on a matrix Lie group G, and show that

$$d\rho + \rho \wedge \rho = 0.$$

Set $\rho = \sum_\alpha \sigma^\alpha T_\alpha$, where the one–forms $\{\sigma^\alpha\}$ span the space of the left-invariant one–forms, and T_α are the generators of the Lie algebra of G which satisfy

$$[T_\alpha, T_\beta] = \sum_\gamma c^\gamma_{\alpha\beta} T_\gamma$$

for some structure constants $c^\gamma_{\alpha\beta}$. Show that

$$d\sigma^\gamma = \sum_{\alpha,\beta} f^\gamma_{\alpha\beta} \sigma^\alpha \wedge \sigma^\beta$$

for some constants $f^\gamma_{\alpha\beta}$ which should be determined.

Find a matrix representation of a 2–dimensional Lie group acting on a real line by $x \to ax + b$, where $(a, b) \in \mathbb{R}^+ \times \mathbb{R}$, and construct the left–invariant one–forms on this group.

3

Let $\mathbf{x} \in \mathbb{R}^3$, and let $\mathbf{E}, \mathbf{B}$ be vector fields on $\mathbb{R}^3$ which do not explicitly depend on t. Show that the trajectories $\mathbf{x}(t)$ of a particle with equations of the motion

$$\ddot{\mathbf{x}} = \mathbf{E} + 2\mathbf{B} \wedge \dot{\mathbf{x}} \tag{1}$$

are unparametrised geodesics of a certain connection (which should be constructed) on $\mathbb{R}^3 \times \mathbb{R}$.

Now assume $\mathbf{E} = -\nabla U, \nabla.\mathbf{B} = 0$. Show that in this case the solution curves of (1) can alternatively be obtained from a Kaluza–Klein reduction of a metric on $\mathbb{R}^5$

$$ds^2 = -2U dt^2 - 4\mathbf{A}.d\mathbf{x}dt + 2dtdu + d\mathbf{x}.d\mathbf{x}$$

along a null isometry $\partial/\partial u$, where $\mathbf{A}$ is some vector potential on $\mathbb{R}^3$.

4

Write an essay on topological degree of maps between manifolds, and its role in field theory.

END OF PAPER