UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 1:30 pm to 4:30 pm

PAPER 55

ADVANCED COSMOLOGY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

In the 3+1 formalism, we represent spacetime using the line element

$$ds^2 = N^2 dt^2 - {}^{(3)}g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,,$$

where ${}^{(3)}g_{ij}(x^i)$ is the three metric on constant time t hypersurfaces Σ, the lapse function $N(t, x^i)$ defines the change in the proper time and the shift vector $N^i(t, x^i)$ gives the change in the spatial coordinates for a 'normal' trajectory defined along $n_\mu = (-N,\ 0,\ 0,\ 0)$. The metric constraint equations and the energy conservation equation are respectively [*Do not attempt to derive these*]:

$$ {}^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi G\rho\,, \tag{1}$$

$$ K^j{}_{i|j} - K_{|i} = 8\pi G \mathcal{J}_i\,, \tag{2}$$

$$ \frac{1}{N}(\dot{\Pi} - N^i\Pi_{|i}) - K\Pi + \frac{N^{|i}}{N}\phi_{|i} - \phi^{|i}{}_{|i} + \frac{dV}{d\phi} = 0\,, \tag{3}$$

for a model with a single scalar field ϕ with $\Pi \equiv n^\mu\partial_\mu\phi = \frac{1}{N}\left(\dot{\phi} - N^i\partial_i\phi\right)$ for the energy-momentum tensor $T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial_\lambda\phi\,\partial^\lambda\phi - V(\phi)\right)$. Here, the intrinsic curvature is ${}^{(3)}R_{ij}$ (with Ricci scalar ${}^{(3)}R$) and the extrinsic curvature is given by

$$K_{ij} \equiv -n_{i;j} = -\frac{1}{2N}\left({}^{(3)}g_{ij,0} - N_{i|j} - N_{j|i}\right)\,,$$

with the trace $K \equiv {}^{(3)}g_{ij}K^{ij}$ and $|$ denotes the covariant derivative in Σ. The matter variables in (1-3) are defined by

$$\rho = n_\mu n_\nu T^{\mu\nu}\,, \qquad \mathcal{J}_i = -n^\mu P^\nu{}_i T_{\mu\nu}\,, \qquad \text{and} \qquad S_{ij} = P^\mu{}_i P^\nu{}_j T_{\mu\nu}\,,$$

where the projection tensor is given by $P^\mu{}_\nu = \delta^\mu{}_\nu - n^\mu n_\nu$.

(i) Evaluate the 3+1 scalar field matter variables to show

$$\rho = \tfrac{1}{2}\Pi^2 + \tfrac{1}{2}\partial_i\phi\,\partial^i\phi + V(\phi)\,, \qquad \mathcal{J}_i = -\Pi\,\partial_i\phi\,,$$

$$S_{ij} = \partial_i\phi\,\partial_j\phi + {}^{(3)}g_{ij}\left[\tfrac{1}{2}(\Pi^2 - \partial_k\phi\,\partial^k\phi) - V(\phi)\right]\,.$$

Specialise to a flat (k=0) FRW model with (background) line element

$$ds^2 = \bar{N}^2(t)\,dt^2 - a^2(t)\,\delta_{ij}dx^i dx^j\,.$$

for a model with a homogeneous and isotropic scalar field $\bar{\phi}(t)$. Show that the extrinsic curvature for this model is given by $K^i{}_j = -H\delta^i{}_j$ where $H = (1/\bar{N})(\dot{a}/a)$. Show that the constraints (1-3) in this case reduce to

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\frac{\dot{\bar{\phi}}^2}{\bar{N}^2} + V(\bar{\phi})\right)\,,$$

$$\ddot{\bar{\phi}} + (3H\bar{N} - \dot{\bar{N}}/\bar{N})\dot{\bar{\phi}} + \bar{N}^2\frac{dV}{d\phi} = 0\,.$$

(ii) Consider linearising about the homogeneous background solution given in part (i) with the scalar field expanded as $\phi = \bar{\phi} + \delta\phi$ and with scalar metric perturbations given by

$$N = \bar{N}(1+\Psi)\,, \qquad N_i = a^2 B_{,i}\,, \qquad {}^{(3)}g_{ij} = a^2[(1-2\Phi)\delta_{ij} + 2E_{,ij}]\,.$$

You may assume that the trace of the linearised intrinsic curvature is ${}^{(3)}R = 4\triangle\Phi$ and the extrinsic curvature becomes

$$K^i{}_j = (-H + \tfrac{1}{3}\kappa)\delta^i_j - (\partial^i\partial_j - \tfrac{1}{3}\triangle\delta^i_j)\chi \ \text{ with } \ \kappa \equiv 3\left(\dot{\Phi}/\bar{N} + H\Psi\right) + \triangle\chi \quad (*)$$

and $\chi = \frac{a^2}{\bar{N}}\triangle(B - \dot{E})$ where the Laplacian is $\triangle\phi = \partial_i\partial^i\phi = \nabla^2\phi/a^2$. Show that the linearised constraint equations (1-3) become

$$\triangle\Phi - H\kappa = 4\pi G\,\delta\rho\,,$$
$$\triangle\chi + \kappa = 12\pi G u\,,$$
$$\frac{\ddot{\delta\phi}}{\bar{N}^2} + \left(3H - \frac{\dot{\bar{N}}}{\bar{N}^2}\right)\frac{\dot{\delta\phi}}{\bar{N}} - \triangle\delta\phi + \frac{d^2V}{d\phi^2}\delta\phi - \frac{\dot{\bar{\phi}}}{\bar{N}}\left(\kappa + \frac{\dot{\Psi}}{\bar{N}} - 3H\Psi\right) + 2\Psi\frac{dV}{d\phi} = 0\,,$$

where you should define $\delta\rho$ and u using the matter and metric perturbations.

(iii) Using $(*)$ and the general linearised energy conservation equation

$$\dot{\delta\rho}/\bar{N} = -3H(\delta\rho + \delta P) + (\bar{\rho} + \bar{P})(\kappa - 3H\Phi) - \triangle u\,,$$

show that the perturbation variable ζ, defined by

$$\zeta \;=\; \Phi \;+\; \frac{1}{3}\frac{\delta\rho}{\bar{\rho} + \bar{P}}\,,$$

is independent of time on superhorizon scales, that is, $\dot{\zeta} = 0$ for long wavelengths $k \ll aH$. [*You may assume a simple equation of state* $P = w\rho$, *with background* $\dot{\bar{\rho}}/\bar{N} = -3H(1+w)\bar{\rho}$.] Briefly give three reasons why ζ is useful for evolving perturbations from early to late cosmological times.

2

(a) Consider a model with a non-canonical kinetic term for which the matter part of the action is given by

$$S_m = \int dtd^3x \sqrt{-g}\, P(X,\phi) \qquad \text{with} \qquad X = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\,,$$

where you may assume that this corresponds to a perfect fluid with pressure P and energy density $\rho = 2XP_{,X} - P$. You are also given that for a flat FRW line element ($\bar{N} = 1$), the following background FRW evolution equations are satisfied (for convenience we take $M_{\rm Pl} = 1$):

$$3H = \rho\,, \quad \dot{H} = -\tfrac{1}{2}(\rho + p) = -XP_{,X}\,, \quad \text{with} \quad \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{XP_{,X}}{H^2}\,.$$

Perturb around a homogeneous and isotropic background field as $\phi(\mathbf{x},t) = \bar{\phi}(t) + \delta\phi(\mathbf{x},t)$, assuming that matter perturbations dominate over metric perturbations, to show that to second-order we have

$$X \;\equiv\; \bar{X} \;+\; \delta X \;\approx\; \tfrac{1}{2}\dot{\bar{\phi}}^2 + \dot{\bar{\phi}}\,\delta\dot{\phi} + \tfrac{1}{2}\delta\dot{\phi}^2 - \tfrac{1}{2}(\partial_i\delta\phi)^2/a^2\,.$$

Transform from this flat gauge to the ζ-gauge using the transformation $\zeta = -(H/\dot{\bar{\phi}})\delta\phi$ to find

$$\delta X \;\approx\; \frac{2\bar{X}}{H}\left(-\dot{\zeta} + \frac{1}{2H}\left(\dot{\zeta}^2 - \frac{(\partial_i\zeta)^2}{a^2}\right)\right)\,.$$

Consider the perturbed matter action

$$S_m = \int dtd^3x\, a^3\left[P_{,X}\delta X + \tfrac{1}{2}P_{,XX}\delta X^2 + \frac{1}{3!}P_{,XXX}\delta X^3\right]\,,$$

and show that the coefficient $\mathcal{C}$ of the $\dot{\zeta}^3$ term in the third-order action is

$$\mathcal{C} = \frac{a^3\epsilon}{H}\frac{(1-c_{\rm s}^2)}{c_{\rm s}^2}\mathcal{A}$$

$$\text{where} \qquad c_{\rm s}^2 = \frac{P_{,X}}{P_{,X} - 2\bar{X}P_{,XX}} \qquad \text{and} \qquad \mathcal{A} = -1 - \tfrac{2}{3}\bar{X}P_{,XXX}/P_{,XX}\,.$$

[*You do not need to evaluate any other terms in the cubic action.*]

(b) According to the in-in formalism during inflation, the leading order correction to an operator Q is given by the expectation value

$$\langle Q(t)\rangle = \mathcal{R}e\left\langle -2iQ^I(t)\int_{-\infty(1-i\mathcal{E})}^{t} H^I_{\rm int}(t')dt'\right\rangle\,, \qquad (\dagger)$$

where we will assume the interaction Hamiltonian $H^I_{\rm int}$ at third-order is given by that found in part (a), that is,

$$H^I_{\rm int} = -\int d^3x\, \frac{a^3\epsilon}{H}\frac{(1-c_{\rm s}^2)}{c_{\rm s}^2}\mathcal{A}\,\dot{\zeta}^3\,, \qquad (\ddagger)$$

with slow-roll parameter ϵ (which you may assume is effectively constant) and scale factor given by $a \approx -1/(H\tau)$ with Hubble constant H and conformal time τ (i.e. $dt = a d\tau$). Here, in the interaction picture, the linear density perturbation ζ is a Gaussian random field with power spectrum,

$$\langle \zeta^I(\mathbf{k},\tau)\zeta^I(\mathbf{k}',\tau)\rangle = (2\pi)^3 u_{\mathbf{k}}(\tau)\, u^*_{\mathbf{k}}(\tau)\, \delta(\mathbf{k}+\mathbf{k}')\,, \qquad (*)$$

where the mode functions $u_{\mathbf{k}}(\tau)$ and their conformal time derivatives are given by

$$u_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon k^3}}\,(1+ik\tau)\,e^{-ik\tau}\,, \qquad u'_{\mathbf{k}}(\tau) = \frac{H}{\sqrt{4\epsilon k^3}} k^2 \tau e^{-ik\tau}\,.$$

(i) Briefly explain Wick's Theorem using the example of a higher-order correlator for a Gaussian random field. Use Wick's theorem, together with the power spectrum $(*)$ and the in-in formalism expression (†), to show that the three point correlator of ζ for the interaction Hamiltonian (‡) reduces to the following terms,

$$\begin{aligned}\langle \zeta(\mathbf{k}_1,0),\zeta(\mathbf{k}_2,0),\zeta(\mathbf{k}_3,0)\rangle \;&=\; \mathcal{R}e\Bigg(-2i\int d\tau \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3} \;\times \\ &\frac{(1-c_{\rm s}^2)}{c_{\rm s}^2}\frac{\mathcal{A}\epsilon}{H(H\tau)}\, u_{\mathbf{k}_1}(0)\, u_{\mathbf{k}_2}(0)\, u_{\mathbf{k}_3}(0)\, u'_{\mathbf{p}_1}(\tau)\, u'_{\mathbf{p}_2}(\tau)\, u'_{\mathbf{p}_3}(\tau) \;\times \\ &(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\left[\delta(\mathbf{k}_1+\mathbf{p}_1)\delta(\mathbf{k}_2+\mathbf{p}_2)\delta(\mathbf{k}_3+\mathbf{p}_3) + \text{cyclic perms.}\right]\Bigg).\end{aligned}$$

(ii) Substitute the mode functions for the density field ζ and evaluate the integrals above explicitly to show that the three-point correlator becomes

$$\langle \zeta(\mathbf{k}_1,0),\zeta(\mathbf{k}_2,0),\zeta(\mathbf{k}_3,0)\rangle = \frac{-3H^4\pi^3}{\epsilon^2}\frac{(1-c_{\rm s}^2)}{c_{\rm s}^2}\mathcal{A}\,\frac{1}{k_1k_2k_3}\frac{1}{K^3}\,\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,,$$

where $K = k_1+k_2+k_3$. Discuss the approximations made and limits taken in evaluating the integral. Briefly comment on whether the non-Gaussian parameter $f_{\rm NL}$ could be detectable for this case.

3

(a) Consider linear scalar perturbations about a spatially-flat Friedmann–Robertson–Walker model with scale factor $a(\eta)$, where η is conformal time. In the Newtonian gauge,

$$ds^2 = a^2(\eta)\left[(1+2\psi)d\eta^2 - (1-2\phi)\delta_{ij}dx^i dx^j\right] .$$

If the components of the 4-momentum of a photon are written as

$$p^\mu = a^{-2}\epsilon\,[1-\psi, (1+\phi)\mathbf{e}] ,$$

with $\mathbf{e}^2 = 1$, show that the energy measured by an observer comoving with the coordinate system is $E = \epsilon/a$.

Use the geodesic equation to show that

$$\frac{d\ln\epsilon}{d\eta} = -\frac{d\psi}{d\eta} + (\dot{\phi} + \dot{\psi}) ,$$

where the derivatives $d/d\eta$ are along the spacetime path of the photon, and overdots denote partial differentiation with respect to η.

[*You may assume the following connection coefficients*:

$$\Gamma^0_{00} = \frac{\dot{a}}{a} + \dot{\psi} , \quad \Gamma^0_{0i} = \partial_i\psi , \quad \Gamma^0_{ij} = \frac{\dot{a}}{a}\delta_{ij} - \left(\dot{\phi} + 2\frac{\dot{a}}{a}(\phi+\psi)\right)\delta_{ij} .$$]

(b) The Boltzmann equation for the fractional temperature anisotropy of the CMB, $\Theta(\eta, \mathbf{x}, \mathbf{e})$, is

$$\frac{\partial\Theta}{\partial\eta} + \mathbf{e}\cdot\boldsymbol{\nabla}\Theta - \frac{d\ln\epsilon}{d\eta} = \dot{\tau}\Theta - \frac{3\dot{\tau}}{4}\int\frac{d\hat{\mathbf{m}}}{4\pi}\,\Theta(\hat{\mathbf{m}})\left[1 + (\mathbf{e}\cdot\hat{\mathbf{m}})^2\right] - \dot{\tau}\mathbf{e}\cdot\mathbf{v}_b ,$$

where $\mathbf{v}_b$ is the baryon peculiar velocity and $\dot{\tau} = -a\bar{n}_e\sigma_{\mathrm{T}}$ is the differential optical depth to Thomson scattering off electrons with background-order electron number density $\bar{n}_e$. Assuming that last scattering is instantaneous at time η_*, so that $-\dot{\tau}e^{-\tau} = \delta(\eta - \eta_*)$ with $\tau = 0$ at the present time η_0, show that the temperature anisotropy observed at η_0 and $\mathbf{x}_0$ satisfies

$$\Theta(\eta_0, \mathbf{x}_0, \mathbf{e}) + \psi(\eta_0, \mathbf{x}_0) \approx (\Theta_0 + \psi + \mathbf{e}\cdot\mathbf{v}_b)(\eta_*, \mathbf{x}_0 - \chi_*\mathbf{e}) + \int_{\eta_*}^{\eta_0} d\eta'\,(\dot{\psi} + \dot{\phi})(\eta', \mathbf{x}_0 - \chi'\mathbf{e}) , \quad (*)$$

where $\chi_* = \eta_0 - \eta_*$, $\chi' = \eta_0 - \eta'$, and Θ_0 is the monopole of $\Theta(\mathbf{e})$. State clearly any further assumptions that you make.

Explain the physical origin of the various terms in $(*)$.

(c) For the remainder of the question, assume that $\phi = \psi$. For adiabatic initial conditions, on super-Hubble scales the gravitational potential is related to the primordial curvature perturbation $\mathcal{R}$ by

$$\mathcal{R} = -\phi - \frac{\mathcal{H}(\dot{\phi} + \mathcal{H}\phi)}{4\pi G a^2(\bar{\rho} + \bar{P})} ,$$

where $\mathcal{H} \equiv \dot{a}/a$, and $\bar{\rho}$ and $\bar{P}$ are the background energy density and pressure, respectively. Given that during radiation domination, Θ_0 is constant on super-Hubble scales and is related to the primordial curvature perturbation $\mathcal{R}$ by $\Theta_0 = \mathcal{R}/3$, show that, in Fourier space,

$$(\Theta_0 + \psi)(\eta_*, \mathbf{k}) = -\mathcal{R}(\mathbf{k})/5$$

for scales that are super-Hubble at η_*.

[*You may assume the photon continuity equation* $\dot{\Theta}_0 + \boldsymbol{\nabla} \cdot \mathbf{v}_\gamma/3 - \dot{\phi} = 0$, *where* $\mathbf{v}_\gamma$ *is the peculiar (bulk) velocity of the photons, and that the universe is matter-dominated at last scattering. You will find it helpful to consider the change in* Θ_0 *and* ψ *through the matter-radiation transition.*]

Hence show that if the integral term is neglected in $(*)$, the angular power spectrum of $\Theta(\eta_0, \mathbf{x}_0, \mathbf{e})$ on large angular scales is approximately

$$C_l \approx \frac{4\pi}{25} \int d\ln k \, \mathcal{P}_{\mathcal{R}}(k) j_l^2(k\chi_*) \qquad (l > 0)\,,$$

where $\mathcal{P}_{\mathcal{R}}(k)$ is the dimensionless power spectrum of $\mathcal{R}$, and the $j_l(x)$ are the spherical Bessel functions. You should explain clearly any further approximations that you make.

[*The plane-wave expansion is*

$$e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi \sum_{lm} i^l j_l(kx) Y_{lm}(\hat{\mathbf{x}}) Y_{lm}^*(\hat{\mathbf{k}})\,.$$
]

4

(a) In the presence of scalar perturbations in the Newtonian gauge, the fractional temperature anisotropy $\Theta(\eta, \mathbf{x}, \mathbf{e})$ of the CMB, at conformal time η, comoving position $\mathbf{x}$, and direction $\mathbf{e}$, satisfies the Boltzmann equation

$$\frac{\partial \Theta}{\partial \eta} + \mathbf{e} \cdot \boldsymbol{\nabla} \Theta = \frac{\partial \phi}{\partial \eta} - \mathbf{e} \cdot \boldsymbol{\nabla} \psi + \dot{\tau} \Theta - \frac{3\dot{\tau}}{4} \int \frac{d\hat{\mathbf{m}}}{4\pi} \Theta(\hat{\mathbf{m}}) \left[1 + (\mathbf{e} \cdot \hat{\mathbf{m}})^2\right] - \dot{\tau} \mathbf{e} \cdot \mathbf{v}_b \,.$$

Here, ϕ and ψ are the gravitational potentials, $\dot{\tau}$ is the differential optical depth, and $\mathbf{v}_b$ is the baryon peculiar velocity. Briefly explain why the Fourier transform $\Theta(\eta, \mathbf{k}, \mathbf{e})$ is axisymmetric about the wavevector $\mathbf{k}$, i.e., one can write

$$\Theta(\eta, \mathbf{k}, \mathbf{e}) = \sum_{l \geqslant 0} (-i)^l \Theta_l(\eta, \mathbf{k}) P_l(\hat{\mathbf{k}} \cdot \mathbf{e}) \,,$$

where the $P_l(\mu)$ are Legendre polynomials.

Use the orthogonality of the Legendre polynomials,

$$\int_{-1}^{1} P_l(\mu) P_{l'}(\mu)\, d\mu = \frac{2}{2l+1} \delta_{ll'} \,,$$

to show that

$$\int \frac{d\hat{\mathbf{m}}}{4\pi} \Theta(\eta, \mathbf{k}, \hat{\mathbf{m}}) \left[1 + (\mathbf{e} \cdot \hat{\mathbf{m}})^2\right] = \frac{4}{3} \Theta_0(\eta, \mathbf{k}) - \frac{2}{15} \Theta_2(\eta, \mathbf{k}) P_2(\hat{\mathbf{k}} \cdot \mathbf{e}) \,.$$

[*You may wish to use* $P_0(\mu) = 1$, $P_1(\mu) = \mu$, *and* $P_2(\mu) = (3\mu^2 - 1)/2$.]

Noting that $(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + l P_{l-1}(\mu)$, derive the Boltzmann hierarchy for the $\Theta_l(\eta, \mathbf{k})$:

$$\dot{\Theta}_l + k \left(\frac{l+1}{2l+3} \Theta_{l+1} - \frac{l}{2l-1} \Theta_{l-1} \right) = -\dot{\tau} \left[(\delta_{l0} - 1)\Theta_l - \delta_{l1} v_b + \frac{1}{10} \delta_{l2} \Theta_2 \right] + \delta_{l0} \dot{\phi} + \delta_{l1} k \psi \,,$$

where overdots denote differentiation with respect to η, and $\mathbf{v}_b(\eta, \mathbf{k}) = i\hat{\mathbf{k}} v_b(\eta, \mathbf{k})$.

(b) Explain briefly what is meant by the tight-coupling approximation.

Use the $l = 2$ moment of the Boltzmann hierarchy to show that

$$\Theta_2(\eta, \mathbf{k}) = -\frac{20}{27} k \dot{\tau}^{-1} \Theta_1(\eta, \mathbf{k}) \qquad (*)$$

to first-order in the tight-coupling approximation.

(c) The linear polarization of the CMB observed at η_0 and $\mathbf{x}_0$ can be approximated by

$$(Q \pm iU)(\eta_0, \mathbf{x}_0, \mathbf{e}) \approx -\frac{\sqrt{6}}{10} \sum_m \Theta_{2m}(\eta_*, \mathbf{x}_0 - \chi_* \mathbf{e})_{\pm 2} Y_{2m}(\mathbf{e}) \,,$$

where η_* is the time of last scattering, $\chi_* = \eta_0 - \eta_*$, and Θ_{2m} are the $l = 2$ multipoles of the fractional temperature anisotropy. Working in Fourier space, with the wavevector along the z-axis (i.e., $\mathbf{k} = k\hat{\mathbf{z}}$), show that for scalar perturbations

$$(Q \pm iU)(\eta_0, k\hat{\mathbf{z}}, \mathbf{e}) \propto \Theta_2(\eta_*, k\hat{\mathbf{z}}) e^{-ik\chi_* \cos\theta} \sin^2\theta \,,$$

where θ is the angle $\mathbf{e}$ makes with the z-axis.

[*The spin-weighted spherical harmomics* ${}_{\pm 2}Y_{20}(\mathbf{e}) \propto \sin^2\theta$.]

By writing the Stokes parameters Q and U in terms of E- and B-mode scalar potentials,

$$Q + iU = \eth\eth(P_E + iP_B)\,, \qquad Q - iU = \bar{\eth}\bar{\eth}(P_E - iP_B)\,,$$

where the spin-raising and lowering operators, $\eth$ and $\bar{\eth}$, are given at the end of the question, show that

$$\frac{d^2}{d\mu^2}(P_E \pm iP_B) \propto \Theta_2(\eta_*, k\hat{\mathbf{z}})e^{-ik\chi_*\mu}\,,$$

where $\mu = \cos\theta$.

Hence show that $P_B = 0$ and

$$P_E(\eta_0, \mathbf{k}, \mathbf{e}) \propto \sum_{l\geqslant 2}(-i)^l(2l+1)\Theta_2(\eta_*, \mathbf{k})\frac{j_l(k\chi_*)}{(k\chi_*)^2}P_l(\hat{\mathbf{k}}\cdot\mathbf{e})\,.$$

[*You may assume the plane-wave expansion* $e^{ix\mu} = \sum_l i^l(2l+1)j_l(x)P_l(\mu)$, *where* $j_l(x)$ *are the spherical Bessel functions.*]

Comment on the implications of the tight-coupling result in $(*)$ for the large-angle polarization.

[*The actions of the spin-raising and lowering operators on a spin-s field* ${}_s\eta(\theta,\phi)$ *are*

$$\eth_s\eta = -\sin^s\theta(\partial_\theta + i\mathrm{cosec}\,\theta\partial_\phi)(\sin^{-s}\theta\,{}_s\eta)\,,$$
$$\bar{\eth}_s\eta = -\sin^{-s}\theta(\partial_\theta - i\mathrm{cosec}\,\theta\partial_\phi)(\sin^s\theta\,{}_s\eta)\,.\quad]$$

END OF PAPER