UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 9:00 am to 12:00 pm

PAPER 54

BLACK HOLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

A spacetime containing a static, spherically symmetric, star has line element

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ .$$

The matter inside the star is described by a perfect fluid with energy momentum tensor $T_{ab} = (\rho+p)u_a u_b + p g_{ab}$ and barotropic equation of state $p = p(\rho)$ with $\rho, p \geqslant 0$, $dp/d\rho > 0$. The Einstein equation reduces to the Tolman-Oppenheimer-Volkov (TOV) equations:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r-2m)}$$

$$\frac{dp}{dr} = -(p+\rho)\frac{(m + 4\pi r^3 p)}{r(r-2m)}$$

(a)(i) Let R denote the radius of the star, so p, ρ vanish for $r > R$. Show that the metric outside the star is the Schwarzschild metric.

(ii) Explain why smooth solutions of the TOV equations form a 1-parameter family, labelled uniquely by $\rho_c \equiv \rho(0)$.

(iii) Assume that the equation of state is known for $\rho \leqslant \rho_0$ but not for $\rho > \rho_0$. Explain why there is a maximum possible mass for the star that is independent of the equation of state for $\rho > \rho_0$.

You may assume that a solution of the TOV equations satisfies

$$\frac{m(r)}{r} < \frac{2}{9}\left[1 - 6\pi r^2 p(r) + (1 + 6\pi r^2 p(r))^{1/2}\right]$$

(b) Now assume that the star has constant density ρ.

(i) Note that $dp/dr < 0$ so the pressure is greatest at the centre of the star. Derive an expression for the pressure at the centre of the star as a function of M (the mass of the star), R and ρ.

(ii) The matter inside the star obeys the dominant energy condition if $p \leqslant \rho$. Derive an upper bound on M/R for a constant density star satisfying this condition.

2

(a) (i) What is a null geodesic congruence? Define the *expansion, rotation* and *shear* of a null geodesic congruence.

(ii) Derive Raychaudhuri's equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \hat{\sigma}^{ab}\hat{\sigma}_{ab} + \hat{\omega}^{ab}\hat{\omega}_{ab} - R_{ab}U^a U^b$$

(b) Any spherically symmetric metric can be written in the form

$$ds^2 = -2f(U,V)dUdV + r(U,V)^2 d\Omega^2$$

where $f(U,V) > 0$ and $r(U,V) > 0$. Assume that $\partial/\partial U$ and $\partial/\partial V$ are future-directed. You may assume that the null energy condition is satisfied.

(i) Prove that $-(dU)^a$ and $-(dV)^a$ are tangent to affinely parameterized null geodesic congruences.

(ii) Calculate the expansion of each of these null geodesic congruences. Explain why the rotation vanishes.

(iii) Let Σ be a Cauchy surface with $r_{,U} < 0$ on Σ. Use Raychaudhuri's equation to derive an equation for $(f^{-1}r_{,U})_{,U}$ and hence prove that $r_{,U} < 0$ in $D^+(\Sigma)$.

(iv) Assume that $D^+(\Sigma)$ contains a trapped 2-sphere $U = U_0$, $V = V_0$. Prove that the 2-sphere $U = U_0$, $V = V_1$ is trapped for any $V_1 > V_0$.

3

A five-dimensional black hole spacetime has metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_3^2$$

where

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4} \qquad 0 < r_- < r_+$$

and

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad 0 \leqslant \chi, \theta \leqslant \pi, \; 0 \leqslant \phi \leqslant 2\pi$$

is the metric of a unit round three-sphere.

(a) Calculate the Komar mass M of this solution, defined by

$$M = \frac{1}{16\pi} \lim_{r\to\infty} \int \star dk$$

where $k = \partial/\partial t$, the integral is taken over a three-sphere of constant t, r, and the orientation is $-dt \wedge dr \wedge d\chi \wedge d\theta \wedge d\phi$.

(b) Show that one can define a quantity $r_\star$ such that $u = t - r_\star$ and $v = t + r_\star$ are constant on outgoing and ingoing radial null geodesics respectively. (You may express $r_\star$ as an integral.)

(c) Starting with $r > r_+$, obtain the metric in ingoing Eddington-Finkelstein coordinates $(v, r, \chi, \theta, \phi)$. Hence show that it can be analytically continued through the surface $r = r_+$ to a new region with $0 < r < r_+$.

(d) Define the *black hole region* of an asymptotically flat spacetime. Prove that the region $r < r_+$ (in ingoing Eddington-Finkelstein coordinates) is within the black hole region, and that the region $r > r_+$ does not intersect the black hole region. [*Hint: show that* r *is non-increasing along any future-directed causal curve with* $r_- < r < r_+$.]

(e) Show that the surface $r = r_+$ is a Killing horizon of k and determine the surface gravity κ.

(f) Assuming that the Penrose diagram of this spacetime is the same as for a non-extreme Reissner-Nordstrom black hole, explain why you would expect the surface $r = r_-$ in the black hole region to be unstable against small perturbations in the region outside the black hole. Why does this support the strong cosmic censorship conjecture?

4

(a) State and prove the version of the first law of black hole mechanics that relates the change in area of the event horizon to the energy and angular momentum of infalling matter. (You may assume Raychaudhuri's equation.)

(b) Consider a massless free scalar field in a spacetime describing spherically symmetric gravitational collapse. Let $\{f_i\}$ denote an orthonormal basis of positive frequency "in" modes, and $\{p_i\}$ an orthonormal basis of positive frequency "out" modes such that

$$p_i = \sum_j \left(A_{ij} f_j + B_{ij} \bar{f}_j\right)$$

Assume that no particles are present near $\mathcal{I}^-$. Prove that the expected number of particles in the ith "out" mode is $(BB^\dagger)_{ii}$. (You may assume standard properties of the Klein-Gordon inner product and commutators of creation and annihilation operators.)

END OF PAPER