UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 9:00 am to 12:00 pm

PAPER 53

COSMOLOGY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a flat FRW universe filled with two non-interacting perfect fluids with equations of state $w_m = 0$ and $w_X = -\frac{1}{3}$. The Friedmann equation is

$$H^2 \equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3} \sum_{i=m,X} \rho_i \ ,$$

where the energy density of each fluid, ρ_i, satisfies

$$\frac{d\rho_i}{dt} = -3H(1+w_i)\rho_i \ .$$

Use t_0 for the present time, H_0 for the Hubble constant and $\Omega_{i,0} \equiv 8\pi G\rho_i(t_0)/3H_0^2$ for the dimensionless density of each component today. The present value of the scale factor is normalized to unity, i.e. $a(t_0) \equiv 1$.

i) Show that the evolution of the conformal Hubble parameter $\mathcal{H} \equiv a'/a$ satisfies

$$2\mathcal{H}' + \mathcal{H}^2 - \alpha^2 = 0 \ ,$$

where primes denote derivatives with respect to conformal time τ, and the constant α should be determined in terms of H_0 and $\Omega_{m,0}$.

[*Hint: Show that* $\rho_X a^2 = const.$]

ii) Show that the parametric solution for the scale factor $a(\tau)$ and the physical time $t(\tau)$ is

$$a(\tau) = \frac{1}{2}\frac{\Omega_{m,0}}{(1-\Omega_{m,0})}\big[\cosh(\alpha\tau) - 1\big] \ ,$$

$$t(\tau) = \frac{H_0^{-1}}{2}\frac{\Omega_{m,0}}{(1-\Omega_{m,0})^{3/2}}\big[\sinh(\alpha\tau) - \alpha\tau\big] \ .$$

iii) Find an expression for the age of this universe t_0 in terms of H_0 and $\Omega_{m,0}$ and show that it satisfies

$$\frac{2}{3}H_0^{-1} \leqslant t_0 \leqslant H_0^{-1} \ ,$$

for all values of $\Omega_{m,0}$.

[*You may use that* $\sinh^{-1} x = x - \frac{1}{6}x^3 + \cdots$ *for* $|x| < 1$.]

2

The early universe was dominated by relativistic species. Per degree of freedom, the number density and energy density of relativistic species are

$$\begin{array}{l} \text{bosons:} \\ \text{fermions:} \end{array} \qquad n = 0.12\,T^3 \left\{ \begin{array}{l} 1 \\ \frac{3}{4} \end{array} \right. \qquad \rho = 3.3\,T^4 \left\{ \begin{array}{l} 1 \\ \frac{7}{8} \end{array} \right. ,$$

where T is the temperature. The Hubble expansion rate then is

$$H = 1.66\,\frac{\sqrt{g_\star}\,T^2}{M_{\rm pl}} ,$$

where $M_{\rm pl} \equiv 1.2 \times 10^{19}$ GeV and $g_\star$ is the effective number of relativistic degrees of freedom.

1) Consider a theory beyond the Standard Model which contains an extra species of massless neutrinos whose interaction rate is given by $\Gamma = G_\nu^2 T^5$, where $G_\nu \approx 10^{-12}$ GeV^{-2}.

i) Estimate the temperature T_{dec} at which these neutrinos decouple from thermal equilibrium.

[*You may assume that all particles of the Standard Model are relativistic at T_{dec}, so that $g_\star(T_{dec}) \sim 100$.*]

ii) What is the present temperature of these neutrinos relative to $T_0 \approx 2 \times 10^{-13}$ GeV, the present temperature of the CMB? Explain your reasoning. What is the present number density of these neutrinos relative to that of the CMB photons?

[*Hint: Make sure to take into account the decoupling of the ordinary neutrinos at $T \sim 1$ MeV.*]

2) Now, let these neutrinos have a mass m_ν.

i) If $T_0 < m_\nu < T_{dec}$, give an estimate for the upper bound on the neutrino mass coming from the requirement that their fractional energy density satisfies $\Omega_\nu \lesssim 0.1$.

[*You may use that the fractional energy density in photons is $\Omega_\gamma \sim 5 \times 10^{-5}$.*]

ii) Discuss qualitatively why a large mass, $m_\nu \gg T_{dec}$, is still compatible with the standard cosmology.

3

Consider a homogenous scalar field, $\phi(t)$ (the 'inflaton'), with potential energy density $V(\phi)$. Its equations of motion in a flat FRW spacetime are

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \ , \qquad H^2 = \frac{1}{3M_{\rm pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right) \ ,$$

where overdots denote derivatives with respect to physical time t, primes denote derivatives with respect to the inflaton, H is the Hubble parameter, and $M_{\rm pl}$ is the reduced Planck mass.

i) Describe qualitatively how quantum fluctuations in the inflaton field lead to fluctuations in the comoving curvature perturbation, $\mathcal{R}$, on super-Hubble scales.

ii) The resulting power spectrum of $\mathcal{R}$ is

$$\Delta_{\mathcal{R}}^2(k) = \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \ ,$$

where the right-hand side is evaluated at $k = aH$.

Define the slow-roll approximation and use it to write the power spectrum in terms of V and $\epsilon \equiv \frac{1}{2}M_{\rm pl}^2(V'/V)^2$.

Show that the spectral index $n_s(k) \equiv 1 + d\ln\Delta_{\mathcal{R}}^2(k)/d\ln k$, is

$$n_s(k) \approx 1 - 6\epsilon + 2\eta \ ,$$

where $\eta \equiv M_{\rm pl}^2 V''/V$.

iii) Now consider the potential $V(\phi) = \frac{1}{2}m^2\phi^2$. Determine the value of m for which this model leads to the observed amplitude of curvature perturbations, $\Delta_{\mathcal{R}}^2(k_\star) = 2.2 \times 10^{-9}$, where $k_\star$ corresponds to a mode that exited the Hubble radius 50 e-folds before the end of inflation.

How does the prediction for $n_s(k_\star)$ compare with the measurement from the Planck satellite, $n_s = 0.96 \pm 0.02$?

iv) Inflation also predicts a stochastic background of gravitational waves with a power spectrum given by

$$\Delta_h^2(k) = \frac{8}{M_{\rm pl}^2}\left(\frac{H}{2\pi}\right)^2 \ .$$

Compute the tensor-to-scalar ratio $r \equiv \Delta_h^2(k_\star)/\Delta_{\mathcal{R}}^2(k_\star)$ for $V(\phi) = \frac{1}{2}m^2\phi^2$. How does it compare to the bound from the Planck satellite, $r < 0.09$?

4

Consider a flat FRW universe filled with cold dark matter (c) and radiation (r). The dark matter density contrast (in comoving gauge) satisfies

$$\Delta_c'' + \frac{a'}{a}\Delta_c' = 4\pi G a^2 (\bar{\rho}_c \Delta_c + 2\bar{\rho}_r \Delta_r) \qquad (\star)$$

where primes denote derivatives with respect to conformal time τ, and $\bar{\rho}_c$ and $\bar{\rho}_r$ are the background densities. Assume that the perturbations are adiabatic on superhorizon scales, i.e. for $k\tau \ll 1$, where k is the comoving wavenumber.

i) Show that outside the horizon the growing mode solution of $(\star)$ is $\Delta_c \propto \tau^2$, in both the radiation-dominated era ($a \propto \tau$) and the matter-dominated era ($a \propto \tau^2$).

[*Hint: First show that adiabatic superhorizon fluctuations satisfy the relation* $\Delta_r = \frac{4}{3}\Delta_c$.]

ii) Describe qualitatively the evolution of $\Delta_r(\tau)$ inside the horizon. Explain why its contribution to the right-hand side of $(\star)$ can then be ignored. Show that the growing mode solution of $(\star)$ in the radiation-dominated era is $\Delta_c \propto \ln \tau$.

iii) Show that in the matter-dominated era Δ_c grows as τ^2 on all scales.

iv) Early in the radiation era, Δ_c has the following spectrum

$$\frac{k^3}{2\pi^2}|\Delta_c(k,\tau)|^2 = A\,\tau^4 k^4 \ , \qquad \text{for} \quad k\tau < 1 \ .$$

Using your previous results, explain why the spectrum retains the same shape in the matter era for modes with $k\tau_{\rm eq} < 1$, but changes for modes with $k\tau_{\rm eq} \gg 1$:

$$\frac{k^3}{2\pi^2}|\Delta_c(k,\tau)|^2 = A\left(\frac{\tau}{\tau_{\rm eq}}\right)^4 (\ln(k\tau_{\rm eq}))^2 \ , \qquad \text{for} \quad k\tau_{\rm eq} \gg 1 \ ,$$

where $\tau_{\rm eq}$ is the time of matter-radiation equality.

Sketch the final form of the spectrum today.

END OF PAPER