UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2015 9:00 am to 12:00 pm

PAPER 52

GENERAL RELATIVITY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

a) Let $(\mathcal{M}, g)$ be a D-dimensional manifold with metric and Levi-Civita connection. Let Σ be a $(D-1)$-dimensional hypersurface of $\mathcal{M}$ and X a vector field that is nowhere tangent to Σ. Briefly describe the construction of coordinates adapted to the integral curves of X.

b) Let $(\mathcal{M}, g)$ be a 4-dimensional spacetime with metric and Levi-Civita connection. The spacetime is defined to be *stationary* if there exists a coordinate chart $\{x^\mu\}$, $\mu = 0, 1, 2, 3$, with timelike coordinate x^0 (i.e. $g_{00} < 0$) such that the metric components satisfy $\partial_0 g_{\mu\nu} = 0$. The spacetime is defined to be *static* if there exists a coordinate chart such that $\partial_0 g_{\mu\nu} = 0$ and $g_{0i} = 0$ where $i = 1, 2, 3$.

(i) Show that the spacetime is stationary if and only if there exists a timelike Killing vector field V.

(ii) Show that if the spacetime is static, there exists a timelike Killing vector field that satisfies $V_{[\alpha}\nabla_\mu V_{\nu]} = 0$.

(iii) Now consider the reverse of question (ii): let V be a timelike Killing vector field with $V_{[\alpha}\nabla_\mu V_{\nu]} = 0$. Show that the condition $V_{[\alpha}\nabla_\mu V_{\nu]} = 0$ implies

$$\nabla_\mu\big(|V|^n V_\nu\big) - \nabla_\nu\big(|V|^n V_\mu\big) = 0\,, \qquad (\dagger)$$

where $|V|^2 := V_\rho V^\rho$ and n is an integer number. Determine n.

(iv) The general solution V_α to Eq. $(\dagger)$ can be expressed in terms of the gradient of a scalar function ϕ. Show that this scalar function is of the form $\phi = x^0 + f(x^i)$ where f is a free function. Use the scalar ϕ to transform from $\{x^\mu\}$ to new coordinates $\{\bar{x}^\alpha\}$ where $\bar{g}_{0i} = 0$ and $\partial_0 \bar{g}_{\mu\nu} = 0$ and thus prove that the metric is static.

2

a) Einstein's first attempt to formulate the field equations of general relativity was $R_{ab} = \kappa T_{ab}$, where R_{ab}, T_{ab} and κ are the Ricci tensor, energy-momentum tensor and a constant, respectively. Argue how this equation is problematic considering conservation of energy-momentum and the contracted Bianchi identities.

b) Let $(\mathcal{M}, g_{ab})$ be a four-dimensional spacetime with Levi-Civita connection. The determinant g of the metric satisfies $\partial g/\partial g_{\alpha\beta} = g\, g^{\alpha\beta}$.

(i) Show that the trace of the Christoffel symbols satisfies

$$\Gamma^{\rho}_{\rho\mu} = \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g}\,.$$

(ii) Show that the two expressions for the energy momentum $T_{\alpha\beta}$ in terms of the matter part S_M of the action

$$T_{\alpha\beta} = -2\frac{\partial L}{\partial g^{\alpha\beta}} + L\, g_{\alpha\beta}\,, \qquad T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L)}{\partial g^{\alpha\beta}}\,,$$

are equivalent. Here L is the Lagrangian in the action

$$S_M = \int_{\mathcal{M}} L(\phi, \phi_{,\alpha}, g_{\alpha\beta})\sqrt{-g}\, d^4x\,.$$

(iii) The matter action of a minimally coupled scalar field is given by

$$S = \int_{\mathcal{M}} \sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) - V(\phi)\right] d^4x\,, \qquad (\dagger)$$

with the potential $V(\phi)$. Derive the equations of motion for the scalar field under the assumption that the variation of the scalar field vanishes on $\partial\mathcal{M}$.

(iv) Calculate the energy momentum tensor $T_{\alpha\beta}$ associated with the action S of Eq. ($\dagger$) and show that this energy momentum tensor obeys the conservation law $\nabla_\mu T^{\mu}{}_{\alpha} = 0$ provided that the equations of motion are satisfied.

3

Let $(\mathcal{M}, g_{\mu\nu})$ be a four-dimensional, globally hyperbolic spacetime with coordinates (x^μ) and the Levi-Civita connection $\Gamma^\mu_{\nu\rho}$. The determinant g of the metric satisfies $\partial g/\partial g_{\alpha\beta} = g\, g^{\alpha\beta}$. The Riemann tensor is given by

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\tau_{\nu\sigma}\Gamma^\mu_{\tau\rho} - \Gamma^\tau_{\nu\rho}\Gamma^\mu_{\tau\sigma}\,.$$

The coordinates (x^μ) are called *harmonic coordinates* if they satisfy

$$\nabla^\mu \nabla_\mu x^\alpha = 0\,, \qquad (\dagger)$$

where the x^α are treated as scalar functions in the covariant dervative.

(i) Briefly comment on whether or not Eq. (†) is a tensorial equation.

(ii) Show that the harmonic gauge condition can be formulated in the following two equivalent forms

$$g^{\beta\gamma}\Gamma^\alpha_{\beta\gamma} = 0 \qquad \Leftrightarrow \qquad \partial_\nu\big(\sqrt{-g}\, g^{\mu\nu}\big) = 0\,.$$

(iii) Show that in harmonic coordinates, the vacuum Einstein equations can be written in the form

$$R_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu}\partial_\mu \partial_\nu g_{\alpha\beta} + \ldots = 0\,,$$

where the dots denote terms (which need not be evaluated explicitly) containing at most first derivatives of the metric components.

(iv) In the "3+1" split of the Einstein equations, the spacetime metric and its inverse are written as

$$g_{\alpha\beta} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^j \\ \alpha^{-2} & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix},$$

where Latin indices range from 1 to 3, γ_{ij} is the spatial metric, γ^{ij} its inverse, β^i the shift vector, $\beta_j = \gamma_{ji}\beta^i$ and α is the lapse function. The ADM equation for the time derivative of the the spatial metric is given by

$$\partial_t \gamma_{ij} - \beta^k \partial_k \gamma_{ij} - \gamma_{kj}\partial_i \beta^k - \gamma_{ik}\partial_j \beta^k = -2\alpha K_{ij}\,.$$

Show that this equation implies

$$\partial_t \gamma - \beta^k \partial_k \gamma - 2\gamma \partial_k \beta^k = -2\alpha\gamma K\,,$$

where $K = \gamma^{mn}K_{mn}$ is the trace of the extrinsic curvature.

(v) The *harmonic slicing* condition is defined by $\nabla^\mu\nabla_\mu x^0 = 0$ where $x^0 = t$ denotes the time coordinate in the 3+1 decomposition. Show that the harmonic slicing condition implies

$$\partial_t \alpha - \beta^i \partial_i \alpha = f(\alpha) K\,,$$

where f is a function of the lapse α. Determine $f(\alpha)$. [You may use that $g = -\alpha^2\gamma$].

4

a) Briefly describe the diffeomorphism invariance of general relativity (formulas need not be given). What is a spacetime in mathematical terms?

b) Let $\mathcal{M}$ be a manifold with coordinate chart (x^μ), metric $g_{\mu\nu}$ and a connection $\Gamma^\alpha_{\mu\nu}$ which need not be the Levi-Civita connection. The torsion tensor is defined as

$$T_{\mu\nu}{}^\lambda = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} \,.$$

We furthermore define the tensor

$$N_{\mu\nu\alpha} = \nabla_\alpha g_{\mu\nu} \,,$$

where ∇ denotes the covariant derivative associated with $\Gamma^\alpha_{\mu\nu}$. The Christoffel symbols are defined as

$$S^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha}\left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}\right) .$$

(i) By expanding the covariant derivative $\nabla_\alpha g_{\mu\nu}$ of the metric, show that

$$\Gamma^\rho_{\mu\nu} = S^\rho_{\mu\nu} - K_{\mu\nu}{}^\rho + W_{\mu\nu}{}^\rho \,,$$

where

$$\begin{aligned} K_{\mu\nu}{}^\rho &= -T_{\mu\nu}{}^\rho - T_\nu{}^\rho{}_\mu + T^\rho{}_{\mu\nu} \,, \\ W_{\mu\nu}{}^\rho &= \frac{1}{2}\left(N_{\mu\nu}{}^\rho - N_\nu{}^\rho{}_\mu - N^\rho{}_{\mu\nu}\right) . \end{aligned}$$

(ii) Briefly interprete the result.

(iii) Let $p \in \mathcal{M}$. The exponential map is defined as

$$e : \mathcal{T}_p(\mathcal{M}) \to \mathcal{M} \,, \quad X_p \mapsto q \,,$$

where q is the point a unit affine parameter distance from p along the geodesic through p with tangent X_p in p. Show that the vector $Y_p := t\, X_p$, $t \in [0,1]$ is mapped under e to the point at affine parameter distance t from p along the same geodesic.

(iv) Show that at the point p in normal coordinates constructed at p

$$S^\lambda_{\mu\nu} + W_{\mu\nu}{}^\lambda - \frac{1}{2} T^\lambda{}_{\mu\nu} - \frac{1}{2} T^\lambda{}_{\nu\mu} = 0 \,.$$

END OF PAPER