UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Tuesday, 9 June, 2015 1:30 pm to 3:30 pm

## PAPER 50

## CLASSICAL AND QUANTUM SOLITONS

*Attempt no more than **TWO** questions.*

*There are **THREE** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

Throughout this paper we define *dimensionless* spacetime coordinates $x_0 = t$ and $x_1 = x$. The sine-Gordon field theory is defined to be the theory of a dimensionless real scalar field $\phi(x,t)$ with Lagrangian density,

$$\mathcal{L} \;=\; \frac{m^2}{\beta^2}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi \,+\, (\cos\phi - 1)\right]$$

where $m$ is a mass scale and $\beta$ is a dimensionless coupling. With these conventions the sine-Gordon equation reads,

$$\partial_\mu\partial^\mu\phi \,+\, \sin\phi \;=\; 0.$$

**1** A family of solutions of the sine-Gordon equation is given as,

$$\phi(x,t) \quad = \quad 4\tan^{-1}\left(\frac{g}{f}\right)$$

with,

$$f(x,t) \quad = \quad \sum_{\mu=0,1}^{(e)} \exp\left[\sum_{i<j}^{N} B_{ij}\mu_i\mu_j \;+\; \sum_{j=1}^{N}\mu_j X_j\right]$$
$$g(x,t) \quad = \quad \sum_{\mu=0,1}^{(o)} \exp\left[\sum_{i<j}^{N} B_{ij}\mu_i\mu_j \;+\; \sum_{j=1}^{N}\mu_j X_j\right]$$

where the $N$ quantities $\mu_i$ each take the values zero and one and the sums $\sum^{(e)}$ and $\sum^{(o)}$ above denote sums over all such values of the $\mu_i$ such that their sum $\sum_{i=1}^{N}\mu_i$ is an even or odd integer respectively. We also have $X_i = \kappa_i x - \beta_i t + \gamma_i$ for $i = 1, 2, \ldots, N$ where $\kappa_i$, $\beta_i$ and $\gamma_i$ are real constants with $\beta_i^2 = \kappa_i^2 - 1$, and,

$$\exp\left(B_{ij}\right) \quad = \quad \frac{(\kappa_i - \kappa_j)^2 - (\beta_i - \beta_j)^2}{(\kappa_i + \kappa_j)^2 - (\beta_i + \beta_j)^2}$$

Note that the quantity on the right hand side of this equation need not be positive.

i) In the case $N = 1$ show that the solution represents a single kink. Determine the velocity $v$ of the kink in terms of the parameters of the solution.

ii) By considering the asymptotic behaviour of the solution in appropriate limits, show that the solution for $N = 2$ corresponds to the scattering of two kinks. Determine the initial velocities $v_1$ and $v_2$ as a function of the parameters of the solution. Find the time delay relative to free motion, $\Delta^{(2)}t[v_1; v_2]$, experienced by the first kink due to the collision.

iii) In the case $N = 3$, the solution corresponds to the scattering of three kinks with velocities $v_1$, $v_2$ and $v_3$ which you should determine. Find the time delay $\Delta^{(3)}t[v_1; v_2, v_3]$ experienced by the first kink and show that,

$$\Delta^{(3)}t[v_1; v_2, v_3] \quad = \quad \Delta^{(2)}t[v_1; v_2] \;+\; \Delta^{(2)}t[v_1; v_3].$$

What is the significance of this relation?

**2** By considering an appropriate matrix element of the time evolution operator $\exp(-i\hat{H}T)$ write down a path integral representation for the exact soliton mass in quantum sine-Gordon theory.

Expand the sine-Gordon field $\phi(x,t)$ in fluctuations around a single kink solution, $\phi_{cl} = 4\tan^{-1}[e^x]$ as $\phi(x,t) = \phi_{cl}(x) + \delta\phi(x,t)$. Show that the classical action $S[\phi]$ appearing in the path integral has an expansion of the form,

$$S[\phi] \quad = \quad -8Tm + \frac{1}{2}\int_0^{Tm} dt \int_{-\infty}^{+\infty} dx\, \delta\phi \left(-\frac{\partial^2}{\partial t^2} - \Delta_x\right)\delta\phi, \;\; + O\left(\delta\phi^3\right)$$

Why is the term linear in $\delta\phi$ absent in this expansion? Find the differential operator $\Delta_x$ explicitly and show that it has a zero eigenvalue, giving the corresponding eigenfunction. What is the significance of this zero mode?

In the following you may assume without proof that, acting on functions defined on the real line, $\Delta_x$ also has a continuum of eigenvalues of the form $\omega^2 = k^2+1$, $k\in\mathbb{R}$ and that the corresponding eigenfunctions $f_\omega(x)$ obey,

$$f_\omega(x) \quad \rightarrow \quad \exp\left(ikx \pm \frac{i}{2}\delta(k)\right),$$

as $x\to\pm\infty$, where $\delta(k)$ is real.

Suppose we now put the system in a large box of length $L \gg 1$ and impose periodic boundary conditions,

$$\delta\phi\,(+L/2) \quad = \quad \delta\phi\,(-L/2)$$

Give without proof a formal expression for the one-loop correction $\Delta M$ to the soliton mass in terms of the eigenvalues, $\omega_n^2$ and $\omega_n^{(0)2}$ of the differential operators $\Delta_x$ and

$$\Delta_x^{(0)} \quad = \quad -\frac{d^2}{dx^2} + 1,$$

subject to these boundary conditions. Here $n$ is an integer label for the resulting discrete spectrum of eigenvalues. Your expression should include the effect of quantum corrections to the vacuum energy, but not of counter-terms.

Show that, for $L \gg 1$,

$$\Delta M \simeq -\frac{m}{2}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\delta(k)\,\frac{k}{\sqrt{k^2+1}}.$$

Briefly explain how the UV divergent part of this expression is cancelled by an appropriate one-loop counter-term.

**3** The two-particle S-matrix for an integrable $O(N)$-invariant field theory in two spacetime dimensions with massive particles in the $N$-dimensional vector representation has the form,

$$S_{ik,jl}(\theta) = \delta_{ik}\delta_{jl}S_1(\theta) + \delta_{ij}\delta_{kl}S_2(\theta) + \delta_{il}\delta_{jk}S_3(\theta)$$

where $\theta$ is the rapidity difference of the incoming particles. Explain how the unknown functions $S_1(\theta)$, $S_2(\theta)$ and $S_3(\theta)$ are constrained by *crossing symmetry* and *unitarity*. (In your discussion of unitarity you may specialize to the $N = 2$ case.)

In this theory the Faddeev-Zamolodchikov algebra takes the form,

$$A_i(\theta_1)A_j(\theta_2) = \delta_{ij}S_1(\theta_{12})\sum_{k=1}^{N} A_k(\theta_2)A_k(\theta_1) + S_2(\theta_{12})A_j(\theta_2)A_i(\theta_1) + S_3(\theta_{12})A_i(\theta_2)A_j(\theta_1)$$

where $\theta_{12} = \theta_1 - \theta_2 > 0$ and $A_i(\theta)$ is an element of the algebra corresponding to a particle of rapidity $\theta$ and $O(N)$ vector index $i$.

Explain what is meant by the *associativity* of this algebra. By imposing associativity in the case $N = 2$, derive the equation,

$$S_2S_1S_3 + S_2S_3S_3 + S_3S_3S_2 = S_3S_2S_3 + S_1S_2S_3 + S_1S_1S_2$$

where the arguments for the three S-matrix factors in each term are $\theta$, $\theta + \theta'$ and $\theta'$ reading from left to right, and the equation holds for all values of the rapidities $\theta$ and $\theta'$.

## END OF PAPER