UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 9:00 am to 12:00 pm

PAPER 5

ANALYSIS OF PARTIAL DIFFERENTIAL EQUATIONS

*There are **THREE** questions in total.*

*Attempt all **THREE** questions.*

The questions carry equal weight of 40 percent.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

This question deals with hyperbolic transport equations.

1. Let us consider the transport equation

$$\begin{cases} \partial_t u + c(t,x)\,\partial_x u = 0, & \forall\, t \geqslant 0,\ x \in \mathbb{R} \\ u(0,x) = u_0(x), & \forall\, x \in \mathbb{R} \end{cases}$$

with $c \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$.

   (a) Recall the definition of weak $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ solutions.

   (b) Let us assume that $c(t,x) := x$ and that u_0 is C^1 on $\mathbb{R}$, then describe the characteristic method and deduce the formula for the classical solutions.

   (c) Show that when u_0 is now merely $L^\infty(\mathbb{R})$, this formula still provides a weak $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ solution, and prove the uniqueness of such solutions.

2. Let us consider (for $a > 0$):

$$\begin{cases} \forall\, x, t \geqslant 0, & \partial_t u + a\partial_x u = 0 \\ \forall\, x \geqslant 0, & u(0,x) = u_0(x) \\ \forall\, t \geqslant 0, & u(t,0) = f(t) \end{cases} \tag{1}$$

where u_0 and f are in $L^\infty(\mathbb{R}_+)$.

   (a) Define the *characteristic curves* (i.e. the curves on which u remains constant) and give their formula, and deduce an explicit formula for the solution (no proof of uniqueness is required here). What conditions on u_0 and f ensures that this solution is a classical solution? What conditions on u_0 and f ensure that $u(t,\cdot) \in C^\infty(\mathbb{R}_+)$ for all $t \geqslant 0$?

   (b) Prove that this formula indeed constructs a weak solution and that such a weak solution is unique.

3. We discuss an exact form for the solution to the Burgers equation and its decay along time. Let f smooth strictly concave with $f(0) = 0$ and $f'(0) = c$, $c \in \mathbb{R}$ and f' bijective on $\mathbb{R}$. We consider the evolution problem

$$\begin{cases} \forall\, t \in \mathbb{R}_+,\ \forall\, x \in \mathbb{R}, & \partial_t u + \partial_x f(u) = 0, \\ \forall\, x \in \mathbb{R}, & u(x,0) = u_0(x) \end{cases}$$

with u_0 smooth and compactly supported with support included in $[-A, A]$.

   (a) Let us denote $t^* \in (0, +\infty]$ the time of existence of a smooth solution. Recall the formula for this time t^* and the explicit formula for the solution up to this time.

   (b) Prove that for all $t \in (0, t^*)$, $u(t,\cdot)$ is a smooth and compactly supported function on $\mathbb{R}$.

   (c) Let us denote $U(t,x) = \int_{-\infty}^{x} u(t,y)\,dy$. Write an evolution equation for U.

(d) Prove that for all $s \in \mathbb{R}$, $t \in (0, t^*)$, $x \in \mathbb{R}$,

$$sf'(s) - f(s) \leqslant \partial_t U(t,x) + f'(s)\partial_x U(t,x).$$

Deduce that if $x(t) := x_0 + tf'(s)$ one has

$$t\left[sf'(s) - f(s)\right] \leqslant U(t,x) - U(0,x_0).$$

(e) Let us denote $h := (f')^{-1}$ and $g(z) := zh(z) - f(h(z))$, $z \in \mathbb{R}$. Prove that

$$U(t,x) = \max_{y\in\mathbb{R}} \left[U(0,y) + tg\left(\frac{x-y}{t}\right)\right].$$

Deduce that $u(t,x) = h((x - x_0)/t)$.
Hint: Prove first that $U(t,x)$ is greater or equal to the right hand side, then prove that the equality is realised when $y = x_0$ by saturating the previous inequality at $s = \partial_x U(t, x(t)) = \partial_x U(0, x_0)$.

(f) Assume that $k_- < h'/2 < k_+$ on $\mathbb{R}$ with $k_- < k_+ < 0$, then prove that

$$\forall\, z \in \mathbb{R}, \qquad \begin{cases} k_-(z-c) \leqslant \dfrac{h(z)}{2} \leqslant k_+(z-c) \\ k_-(z-c)^2 \leqslant g(z) \leqslant k_+(z-c)^2. \end{cases}$$

(g) Let us denote for $t \in (0, t^*)$ and $x \in \mathbb{R}$:

$$G_{t,x}(y) := \int_{-\infty}^{y} u_0(z)\,\mathrm{d}z + tg\left(\frac{x-y}{t}\right).$$

Prove that

$$-\|u_0\|_{L^1} \leqslant \max_{y\in\mathbb{R}} G_{t,x}(y) \leqslant \|u_0\|_{L^1} + \frac{k_+}{t}(x - x_0 - ct)^2.$$

Deduce that

$$\left|\frac{x-x_0}{t} - c\right| \leqslant \sqrt{\frac{2\|u_0\|_1}{-k_+ t}},$$

and finally

$$|u(t,x)| \leqslant \frac{K}{\sqrt{t}}, \quad K := -2k_-\sqrt{\frac{2\|u_0\|_1}{-k_+}}.$$

2

This question deals with complex Hilbertian analysis, compact operators and the original proof of Weyl's theorem. We recall that a (complex) Banach space is a complete normed real vector space, and a (complex) Hilbert space is a separable (real) Banach space whose norm derives from a Hermitian inner product.

1. Prove that a Hilbert space is finite-dimensional if and only if its unit ball is strongly compact (i.e. compact in the topology induced by the norm).
   Hint: Prove that if the dimension is infinite there exists an infinite sequence f_n in the unit ball so that $\|f_n - f_m\| = 1$ for any $m \neq n$, by using an orthornormalisation process.

2. Prove that a Banach space is finite-dimensional if and only if its unit ball is strongly compact.
   Hint: Prove that if the dimension is infinite there exists an infinite sequence f_n in the unit ball so that $\|f_n - f_m\| = 1/2$ for any $m \neq n$, by using an "almost orthonormalisation" process.

3. Prove that the unit ball of a Hilbert space H is compact for the weak topology, i.e. any bounded sequence f_n of H has a subsequence $f_{\varphi(n)}$ (φ increasing from $\mathbb{N}$ to $\mathbb{N}$) and $g \in H$ so that

   $$\forall\, h \in H, \quad \langle f_{\varphi(n)}, h \rangle \xrightarrow[n \to +\infty]{} \langle g, h \rangle.$$

   Hint: Use an Hilbertian base and the Cantor diagonal argument.

4. We consider from now on a Hilbert space H and a *bounded operator* L (i.e. a continuous linear application from H to H). The *spectrum* $\Sigma(L)$ is defined as the set of $\lambda \in \mathbb{C}$ so that $(L - \lambda)$ is not invertible. Are necessarily all element of $\Sigma(L)$ eigenvalues, i.e. so that there is $f \in H$ with $Lf = \lambda f$? Give a proof or a counter-example.

5. We say that the bounded operator L is self-adjoint if

   $$\forall\, f, g \in H, \quad \langle Lf, g \rangle = \langle f, Lf \rangle.$$

   Prove that for such an operator $\Sigma(L) \subset \mathbb{R}$.

6. If L is a bounded self-adjoint operator and $\mathrm{Ker}(L) = \{0\}$ prove that image is dense in H.

7. If L is a bounded self-adjoint operator prove that $\lambda \in \Sigma(L)$ if and only if there exists a sequence $f_n \in H$ with $\|f_n\| = 1$ and $(L - \lambda) f_n \to 0$.

8. If L is a bounded self-adjoint operator and $\mathrm{Ker}(L)$ is finite-dimensional does the range needs being closed? Justify your answers.

9. For L bounded self-adjoint operator on H, we define the *discrete spectrum* $\Sigma_d(L) \subset \Sigma(L)$ as the set of $\lambda \in \mathbb{R}$ so that $\mathrm{Ker}(L - \lambda)$ is finite-dimensional and different from $\{0\}$, and $\mathrm{Im}(L)$ is closed. The *essential spectrum* $\Sigma_e(L)$ is the remainder $\Sigma(L) \setminus \Sigma_d(L)$. Prove that $0 \notin \Sigma_e(L)$ if and only if every bounded sequence f_n of H with Lf_n converging has a convergent subsequence.

10. Prove that $\lambda \in \Sigma_e(L)$ if and only if there exists a sequence $f_n \in H$ with $\|f_n\| = 1$, f_n weakly converging to zero in H and $(L - \lambda)f_n \to 0$.

11. The bounded operator K is said *compact* if it maps the unit ball to a set whose closure is compact (for the strong topology). Prove that a bounded operator K is compact if and only if: for any f_n weakly converging in H, then Kf_n is strongly converging.
Hint: You can use without a proof here the fact (uniform boundedness principle) that a weakly converging subsequence is bounded.

12. Let L be a self-adjoint bounded operator on H, and K be compact and self-adjoint bounded operator on H, prove that $\Sigma_e(L) = \Sigma_e(L + K)$.

3

This question deals with the regularity of elliptic equations with rough coefficients, and follows ideas due to De Giorgi, Nash and Moser. We consider the following equation for $u \geqslant 0$

$$\nabla_x \cdot (A(x)\nabla_x u) = 0, \quad x \in B(0,1) \subset \mathbb{R}^\ell,\ \ell \geqslant 1 \tag{1}$$

where $B(0,1)$ is the standard open ball, $A = A(x)$ is a space-dependent symmetric matrix assumed to be a measurable function of x and such that

$$\forall\, x \in B(0,1), \quad \lambda\, \mathrm{Id} \leqslant A(x) \leqslant \Lambda\, \mathrm{Id}$$

with Id the identity matrix, and $0 < \lambda < \Lambda < +\infty$.

1. We start with a preliminary study of the H^1 space in dimension one.

   (a) Recall the definition of the space $H^1((-a,a))$ on an open interval $(-a,a)$, $a > 0$.
   Hint: Do not forget that it is a subspace of $L^2((-a,a))$ made of classes of equivalence for the relation of equivalence of being equal almost everywhere.

   (b) Recall why any $u \in H^1((-a,a))$ is the limit in H^1 and almost everywhere of a sequence u_n of smooth functions on $(-a,a)$.

   (c) Prove that any $u \in H^1((-a,a,))$ satisfies

   $$\text{ess sup}_{x \neq y \in (-a,a)} \frac{|u(x) - u(y)|}{|x-y|^{1/2}} \leqslant \|u\|_{H^1((-a,a))}$$

   where "ess sup" mean the essential supremum, i.e. for almost every $x, y \in (-a,a)$, $x \neq y$.
   Hint: Use approximation and the fundamental theorem of calculus.

   (d) Prove that any $u \in H^1((-a,a,))$ satisfies

   $$\|u\|_{L^\infty(-a,a)} = \text{ess sup}_{x \in (-a,a)} |u(x)| \leqslant \|u\|_{H^1((-a,a))}$$

   where "ess sup" mean the essential supremum, i.e. for almost every $x \in (-a,a)$.
   Hint: Use approximation, the fundamental theorem of calculus and the point x_0 where a smooth function on $(-a,a)$ is equal to its average.

2. When the dimension is one $\ell = 1$ (therefore $A(x)$ is a real-valued measurable function), we shall prove the following estimate:

$$\text{ess sup}_{x \in (-1/2,1/2)} |u(x)| + \text{ess sup}_{x \neq y \in (-1/2,1/2)} \frac{|u(x) - u(y)|}{|x-y|^{1/2}} \leqslant C\left(\frac{\Lambda}{\lambda}\right) \|u\|_{L^2((-1,1))} \tag{2}$$

   for some constant $C(\Lambda/\lambda)$ depending only on the ratio Λ/λ.

(a) Consider a smooth function ζ that is equal to one on $(-1/2, 1/2)$ and zero outside $(-1, 1)$, and prove that any smooth solution to (1) satisfies

$$\int_{(-1/2,1/2)} u'(x)^2 \,\mathrm{d}x \leqslant 4 \left(\frac{\Lambda}{\lambda}\right)^2 \int_{(-1,1)} u(x)^2 |\zeta'(x)|^2 \,\mathrm{d}x.$$

(b) Deduce the estimate (2) by combining the previous section and approximation arguments.

3. In general dimension $\ell \geqslant 2$, we shall prove the following weaker result of gain of integrability only (the gain of Hölder regularity is also true but will not be considered here):

$$\text{ess sup}_{x \in B(0,1/2)} |u(x)| \leqslant C\left(\ell, \frac{\Lambda}{\lambda}\right) \|u\|_{L^2(B(0,1))} \tag{3}$$

for some constant $C(\ell, \Lambda/\lambda)$ depending only on the dimension ℓ and the ratio Λ/λ.

(a) For a smooth function ζ that is equal to one on $B(0, r_1)$ and zero outside $B(0, r_2)$, $0 < r_1 < r_2 \leqslant 1$, prove that any smooth solution to (1) satisfies

$$\int_{B(0,r_1)} |\nabla_x u(x)|^2 \,\mathrm{d}x \leqslant 4 \left(\frac{\Lambda}{\lambda}\right)^2 \int_{B(0,r_2)} u(x)^2 |\nabla_x \zeta(x)|^2 \,\mathrm{d}x.$$

(b) Deduce that there exists $\alpha > 1$ and a constant $C(\ell)$ depending on the dimension so that

$$\|u\|_{L^{2\alpha}(B(0,r_1))} \leqslant \frac{C(\ell)}{(r_2 - r_1)} \left(\frac{\Lambda}{\lambda}\right) \|u\|_{L^2(B(0,r_2))}.$$

Hint: You can use the following Sobolev embedding result on $B(0, 1/2)$: for any $p \in [2, p^*)$ there is a constant $C(\ell, p)$ so that

$$\|u\|_{L^p(B(0,1/2))} \leqslant C(\ell, p) \|u\|_{H^1(B(0,1/2))}$$

with $p^* = +\infty$ in dimension $\ell = 2$ and $p^* = 2\ell/(\ell - 2)$ else.

(c) Prove that if u is a smooth solution to (1) then for any $q \geqslant 2$, the previous estimate can be performed on $u^{q/2}$ (with an additional term *which has the good sign*!) to get

$$\int_{B(0,r_1)} |\nabla_x u^{q/2}(x)|^2 \,\mathrm{d}x \leqslant 4 \left(\frac{\Lambda}{\lambda}\right)^2 \int_{B(0,r_2)} u(x)^q |\nabla_x \zeta(x)|^2 \,\mathrm{d}x$$

and finally

$$\|u\|_{L^{q\alpha}(B(0,r_1))} \leqslant \frac{C(\ell)}{(r_2 - r_1)^{2/q}} \left(\frac{\Lambda}{\lambda}\right)^{2/q} \|u\|_{L^q(B(0,r_2))}.$$

(d) (Hard question) By an iteration prove the desired estimate (3).
Hint: Use the sequence of Lebesgue exponents and radii $q_i = 2\alpha^i$ and $r_i = 1/2 + 1/2^{i+1}$, $i \geqslant 0$.

END OF PAPER