UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Wednesday, 3 June, 2015   9:00 am to 12:00 pm

PAPER 49

STRING THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

The closed Nambu-Goto string of tension T, in a four-dimensional Minkowski spacetime with cartesian coordinates $\{X^m; m = 0,1,2,3\}$, has phase-space action

$$I[X,P;e,u] = \int dt \oint d\sigma \left\{ \dot{X}^m P_m - \frac{1}{2} e \left[P^2 + (TX')^2 \right] - u X'^m P_m \right\} ,$$

where (e,u) are Lagrange multipliers for constraints. Find the equations of motion for (X,P). Explain *briefly* why the constraints are associated to gauge invariances of the action. [*You should not attempt to derive the gauge transformations or compute Poisson brackets.*]

Show that the phase-space action is equivalent to the Nambu-Goto action $I_{\rm NG}[X]$ given by the area of the worldsheet in the induced metric. By varying $I_{\rm NG}[X]$, or otherwise, find the Nambu-Goto equations of motion for $X^m(t,\sigma)$. Show that these equations are solved by the following closed string configuration (for positive constant R):

$$X^0 = Rt\,, \qquad X^3 = 0\,, \qquad X^1 + iX^2 = (R\cos t)\, e^{i\sigma}\,.$$

Describe the motion. What is the proper length of the string? What is its total energy?

For a string that is open rather than closed, show that the boundary conditions at the ends of the string must be such that for any variation $\delta\vec{X}$ of the space components of X,

$$\left(\vec{X}' \cdot \delta\vec{X}\right)_{\rm ends} = 0\,, \qquad (\vec{X}' = \partial_\sigma \vec{X}).$$

[*It may be assumed that X^0 and e are free variables at the endpoints, except for the restriction $e \neq 0$.*] This allows free-end boundary conditions, in which case

$$X'|_{\rm ends} = 0\,.$$

These boundary conditions are manifestly spacetime-translation and Lorentz invariant; write down the respective Noether charges $\mathcal{P}_m$ and $\mathcal{J}_{mn}$ and verify that they are constants of the motion.

Describe briefly the other possible open-string boundary conditions.

2

A free spin-2 particle of non-zero mass m is described by a symmetric tensor field h_{mn} satisfying the equations

$$\left(\Box_D - m^2\right) h_{mn} = 0\,, \qquad \partial^m h_{mn} = 0\,, \qquad \eta^{mn} h_{mn} = 0\,. \qquad (*)$$

The last two of these equations are called the "subsidiary conditions". By choosing light-cone coordinates $m = (+,-,I)$ and assuming that ∂_- is invertible, show that the subsidiary conditions can be solved for all but $(D-2)(D+1)/2$ components of h_{mn}. What is their $SO(D-2)$ representation content. Which tensor of $SO(D-1)$ has this $SO(D-2)$ decomposition?

In light-cone gauge, the open Nambu-Goto string with free ends in a D-dimensional Minkowski spacetime has an action that can be put in the form

$$I[x,p,\boldsymbol{\alpha}_k;e_0] = \int dt \left\{ \dot{x}^m p_m + \sum_{k=1}^{\infty} \frac{i}{k} \boldsymbol{\alpha}_{-k} \cdot \dot{\boldsymbol{\alpha}}_k - \frac{1}{2} e_0 \left(p^2 + \mathcal{M}^2\right) \right\} ,$$

where $\{x^m, p_m; m = 0,1,\ldots D-1\}$ are the spacetime position and D-momentum of the centre of mass. What is the significance of the variables $\boldsymbol{\alpha}_k$ and how does $\mathcal{M}^2$ depend on them? Explain briefly how your result leads to an organisation of the mass spectrum of the quantum string according to a non-negative integer level number N. Write down the light-cone-gauge states of the open Nambu-Goto string at levels $N = 0,1,2$. Explain why the $N=1$ states must be massless. How are equations $(*)$ relevant to the $N=2$ states?

For the Neveu-Schwarz sector of the open spinning string with free ends, light-cone gauge quantization leads to an organisation of the mass spectrum according to a level N such that $2N$ is a non-negative integer. Define the oscillator vacuum and hence show that the first excited states (at level $N = 1/2$) are

$$b^I_{-\frac{1}{2}}|0\rangle\,,$$

where you should explain the significance of the operator appearing in this expression. Why must these states have zero mass? The massive states at level $N=1$ are

$$\alpha^I_{-1}|0\rangle\,, \qquad b^I_{-\frac{1}{2}} b^J_{-\frac{1}{2}}|0\rangle\,.$$

What are these representations of $SO(D-2)$? Why are they *not* the same as those relevant to the massive spin-2 particle. How can they be assembled to form a tensor of $SO(D-1)$?

3

Write the Nambu-Goto string action in Polyakov form $I[X;\gamma]$, where γ is the independent worldsheet metric. Show that elimination of the momentum variable P from the phase-space action results in the Polyakov action with γ expressed in terms of the Lagrange multipliers for the Hamiltonian constraints.

Define the conformal gauge in both the Hamiltonian and Polyakov formulations of the Nambu-Goto string, and show that they are equivalent. Explain why the conformal gauge choice leaves a residual invariance corresponding to conformal isometries of 2D Minkowski space.

Implementation of gauge conditions in the path-integral approach to quantization leads to an extension of the classical action to include Faddeev-Popov (anti)ghosts. State, without proof, the prescription for finding this action, and apply it to deduce the FP ghost action for the closed Nambu-Goto string. Explain *briefly* its relevance for cancellation of the conformal anomaly. [*You may use without proof the fact that the Virasoro algebra for a conformally-invariant bc-ghost system has central charge* $-2(6J^2 - 6J + 1)$ *when* b *has conformal dimension* J.]

In units for which $2\pi T \equiv 1/\alpha' = 1$, the Veneziano amplitude for the scattering of two open-string tachyons is

$$A(s,t) = \frac{\Gamma(-1-s)\Gamma(-1-t)}{\Gamma(-2-s-t)},$$

where (s,t) are Mandelstam variables. Explain briefly the physical significance of these variables? Find the poles of A as a function of s for fixed t. Use your result to show that there is a massless spin-1 bound state of two tachyons .

4

Write an essay on the "old covariant" approach to quantization of the Nambu-Goto string. Your essay may be restricted to the open string with free ends and should cover the following items:

- Why it is inconsistent to impose all constraints as physical state conditions.
- How absence of level-1 ghosts restricts the intercept parameter a.
- How equivalence with light-cone gauge results is possible at level 1.
- How the critical dimension emerges at level 2.

END OF PAPER