UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 1:30 pm to 4:30 pm

## PAPER 47

## THE STANDARD MODEL

*Attempt no more than* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $\hat{C}$ be the unitary operator corresponding to charge conjugation. Take as given that positive and negative frequency spinors, $u^s(p)$ and $v^s(p)$ respectively, are related by

$$v^s(p) = C\bar{u}^{sT}(p) \quad \text{and} \quad u^s(p) = C\bar{v}^{sT}(p)$$

where $C$ is a $4 \times 4$ spin-matrix and $T$ denotes the transpose. Show that, for a Dirac field $\psi(x)$,

$$\hat{C}\psi(x)\hat{C}^{-1} = \eta_C C\bar{\psi}^T(x)\,.$$

Given that $C$ is a unitary matrix with the property that

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu$$

show that

$$[C^T C^{-1}, \gamma^\mu] = 0$$

and argue that this implies $C^T = \pm C$. For the remainder of the problem, you may take as given $C^T = -C$.

Show that if $\psi(x)$ satisfies the Dirac equation, so does $\psi^c(x) \equiv \eta_C C\bar{\psi}^T$.

Show that $\hat{C}\psi_L(x)\hat{C}^{-1}$ has the same chirality as $\psi_L$.

In just a few paragraphs, explain with a simple example why both C and CP violation are necessary conditions for baryogenesis to occur in the early universe. [Here C stands for charge conjugation and CP stands for the product of charge conjugation with parity.]

**2**

Consider $SU(2)$ gauge theory coupled to a 2-component complex scalar field $\phi$

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a,\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) - \frac{\lambda}{2}\left(\phi^\dagger\phi - \frac{v^2}{2}\right)^2$$

with $\lambda > 0$ and $v^2 > 0$. The covariant derivative is $D_\mu\phi = \partial_\mu\phi + igA^a_\mu\tau^a\phi$, with the generators related to the Pauli $\sigma$-matrices $\tau^a = \sigma^a/2$ and the field strength tensor $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc}A^b_\mu A^c_\nu$.

Identify the relevant degrees-of-freedom and determine their masses (ignoring quantum corrections). Comment on the number of massless fields based on symmetry principles.

What are the interaction terms involving the scalar field?

Explicitly determine any terms in $\mathcal{L}$ which give rise to gauge-boson self-interactions and sketch the vertices.

**3**

Attempt to answer all of the following parts. At the end of the question (continued on the next page) a number of expressions are given which you may use without proof.

(a) For each of the following elastic scattering processes, draw all of the relevant tree-level, Standard Model Feynman diagrams:

$$\text{(i) } \nu_\mu e^- \to \nu_\mu e^- \quad \text{(ii) } \nu_e e^- \to \nu_e e^- \quad \text{(iii) } \bar{\nu}_e e^- \to \bar{\nu}_e e^-$$

(b) Treating the neutrinos as strictly massless, and working in the limit where the electron mass can be neglected at leading order, use the effective weak Lagrangian

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} \sum_{\ell \in \{e,\mu\}} \Big[ \bar{\nu}_\ell \gamma^\alpha (1-\gamma^5) e \, \bar{e} \gamma_\alpha (1-\gamma^5) \nu_\ell \\ + \bar{\nu}_\ell \gamma^\alpha (1-\gamma^5) \nu_\ell \, \bar{e} \gamma_\alpha (c_V - c_A \gamma^5) e + \dots \Big]$$

(the omitted terms are not useful here) to calculate the scattering amplitude $\mathcal{M}(\nu_\mu e)$ for process (i), given the following labels for each particle's momentum: $\nu_\mu(k) + e^-(p) \to \nu_\mu(k') + e^-(p')$. You may assume that $c_V$ and $c_A$ are real.

Show the total cross-section

$$\sigma(\nu_\mu e^- \to \nu_\mu e^-) = G_F^2 F(s)(c_V^2 + A c_V c_A + B c_A^2 + C)$$

where $F(s)$ is a function of $s = (p+k)^2$ and $A$, $B$, and $C$ are constants, all of which you should determine. [Hint: To simplify the integration, work in the centre-of-momentum frame, let $\theta$ be the angle between $\vec{p}$ and $\vec{p}\,'$ and apply momentum conservation even before integrating. This should result in an amplitude-squared with terms proportional to $s^2 f(\cos\theta)$, where $f(\cos\theta)$ is a polynomial in $\cos\theta$.]

(c) Consider processes (ii) and (iii) now. Write down the scattering amplitudes, $\mathcal{M}(\nu_e e)$ and $\mathcal{M}(\bar{\nu}_e e)$, for $\nu_e(k) + e^-(p) \to \nu_e(k') + e^-(p')$ and $\bar{\nu}_e(k) + e^-(p) \to \bar{\nu}_e(k') + e^-(p')$ respectively. How are these related to $\mathcal{M}(\nu_\mu e)$? Without doing much of the work necessary for part b, use these relations to write down the total cross sections for (ii) and (iii). You will find it helpful to use the following Fierz identity relating products of matrix elements:

$$[\gamma^\alpha(1-\gamma^5)]_{ij}[\gamma_\alpha(1-\gamma^5)]_{kl} = -[\gamma^\alpha(1-\gamma^5)]_{il}[\gamma_\alpha(1-\gamma^5)]_{kj}\,.$$

[The following expressions may be used without proof:

$$\begin{aligned} \mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd} \\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ \mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4i\epsilon^{\mu\nu\rho\sigma} \\ \epsilon^{\alpha\beta\sigma\rho}\epsilon_{\alpha\beta\lambda\tau} &= -2(\delta^\sigma_\lambda\delta^\rho_\tau - \delta^\sigma_\tau\delta^\rho_\lambda) \end{aligned}$$

(The overall sign of the last equation was incorrect in the lecture notes, but is correct here.) The differential cross-section for a decay $i \to f$ is given by

$$d\sigma = \frac{1}{|\vec{v}_A - \vec{v}_B|} \frac{1}{4E_A E_B} |\mathcal{M}_{fi}|^2 \, d\rho_f$$

$$d\rho_f = (2\pi)^4 \delta^{(4)} \Big( p_i - \sum_{r \in f} p_r \Big) \prod_{r \in f} \left( \frac{d^3 p_r}{(2\pi)^3 2p_r^0} \right) \quad ]$$

**4**

(a) Consider the coupling of Standard Model fermions to a scalar field $\phi$ via terms such as

$$-\sqrt{2}\lambda_{ij} \bar{\psi}_L^i \phi \, \psi_R^j \, + \text{h.c.}$$

Explain how the fields $\psi_L$, $\psi_R$, and $\phi$ transform under $SU(2)_L \times U(1)_Y$ gauge transformations, including a table of representations and hypercharges for the Standard Model fermions and scalar. What do the indices $i$ and $j$ correspond to?

(b) For each of the following decays or transitions, *briefly* discuss whether they are allowed in the Standard Model and, if so, via what term(s) in the Lagrangian.

(i) $b \to u\, e^- \bar{\nu}_e$ (ii) $\bar{s}\, d \to \bar{d}\, s$ (iii) $h \to W^+ W^-$ (iv) $h \to \tau^+ \mu^-$

(c) Assume that experiments measure a much larger decay rate for some process than expected according to Standard Model calculations. Write a paragraph or two describing how we might modify the Standard Model Lagrangian to account for new physics.

Let us imagine that the experimentally observed enhancement is for the decays $h \to \tau^+ \mu^-$ and $h \to \tau^- \mu^+$. Show that by adding to the Lagrangian the term

$$\Delta L = -\frac{\sqrt{2}\lambda'_{ij}}{\Lambda^2} (\bar{\psi}_L^i \phi \, \psi_R^j)(\phi^\dagger \phi) \, + \text{h.c.}$$

we can describe this enhancement.

**END OF PAPER**