UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 1:30 pm to 4:30 pm

PAPER 46

ADVANCED QUANTUM FIELD THEORY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a Euclidean scalar field theory with scale Λ effective action

$$S_{\Lambda}^{\text{eff}}[\phi] = \int d^4x \left(\frac{1}{2}\partial^{\mu}\phi\, \partial_{\mu}\phi + \frac{1}{2}m^2\phi^2 + \frac{1}{4!}g\phi^4 \right) .$$

The partition function $\mathcal{Z}(m^2, g)$ is defined by

$$\mathcal{Z}(m^2, g) \equiv \int_{\leqslant \Lambda} \mathcal{D}\phi \, \exp\left(-S_{\Lambda}^{\text{eff}}[\phi]\right)$$

where the subscript on the path integral indicates an integral over only those Fourier modes $\tilde{\phi}(p)$ of ϕ with $|p| \leqslant \Lambda$.

a) How is the effective action $S_{\Lambda'}^{\text{eff}}[\phi]$ at a lower scale $\Lambda' < \Lambda$ defined? Show that

$$S_{\Lambda'}^{\text{eff}}[\phi] = S_{\Lambda}^{\text{eff}}[\phi] - \ln\left[\int_{\Lambda' < p \leqslant \Lambda} \mathcal{D}\hat{\phi} \, \exp\left(-\Delta S[\phi, \hat{\phi}]\right)\right] \tag{1}$$

where the meaning of $\hat{\phi}$ should be explained. Find an expression for $\Delta S[\phi, \hat{\phi}]$.

b) Draw all the Feynman diagrams that contribute to $S_{\Lambda'}^{\text{eff}}[\phi]$ up to and including order g^2, and identify the vertices that each of these contribute to. [You are not required to evaluate the diagrams.] What vertex types do you expect $S_{\Lambda'}^{\text{eff}}[\phi]$ to contain if one could evaluate (1) to all orders in g?

c) Now consider beginning with a generic effective action at scale Λ, containing vertices involving arbitrary powers of ϕ and its derivatives. We lower the cutoff to $\Lambda' = b\Lambda$ with $b < 1$, then rescale all lengths as $x \to x' = bx$ in position space, and finally restore the kinetic term to its canonical form. Show that the (dimensionful) coupling κ of a generic vertex involving n powers of ϕ and m derivatives becomes

$$\kappa' = \frac{\kappa + \Delta\kappa}{(1 + \Delta Z)^{n/2}} \left(\frac{\Lambda'}{\Lambda}\right)^{m+n-4} ,$$

where the meaning of $\Delta\kappa$ and ΔZ should be explained.

2 Consider the cubic scalar field theory

$$S[\phi] = \int d^6x \left(\frac{1}{2}\partial^\mu \phi \, \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{6}\phi^3 \right)$$

in *six* Euclidean dimensions. In dimensional regularization, the 1-loop correction to the momentum space propagator of the scalar field is given by the integral

$$I(k) \equiv \frac{(-g)^2 \mu^{6-d}}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p+k)^2 + m^2} \frac{1}{p^2 + m^2} \tag{1}$$

where k is the momentum carried by the incoming scalar.

(i) With the help of a Feynman diagram, explain the origin of each of the different factors that appear in (1).

(ii) Show that the divergent part of the integral in equation (1) is

$$-\frac{1}{\epsilon} \frac{g^2}{(4\pi)^3} \left(m^2 + \frac{k^2}{6} \right)$$

in $d = 6 - \epsilon$ dimensions.

(iii) Which counterterms are necessary to absorb this divergence? Find the values of these counterterms to 1-loop accuracy in the minimal subtraction scheme.

[In this question, you may use without proof the Feynman trick

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[Ax + B(1-x)]^2},$$

and that the volume of a $d-1$ dimensional unit sphere is $\mathrm{Vol}(S^{d-1}) = (2\pi^{d/2}) \,/\, \Gamma(d/2)$, where the Gamma function is defined by the integral $\Gamma(t) = \int_0^\infty ds \, s^{t-1} \, \mathrm{e}^{-s}$. You many also use the fact that $\Gamma(\epsilon/2 - 1) = -\frac{2}{\epsilon} + \text{finite as } \epsilon \to 0$.]

3

The dynamics of a (Euclidean) quantum mechanical system can be described by the path integral

$$\int \mathcal{D}p\, \mathcal{D}q\ \exp\left(\mathrm{i}\int_0^T dt\, p\dot{q} - \int_0^T dt\, H(p,q)\right)\ , \tag{1}$$

with an action that is first–order in the time derivative of $q(t)$ and involves the Hamiltonian $H(p,q)$.

(i) Give a precise definition of the meaning of the derivative $\dot{q}$ in the regularized path integral.

(ii) In the case that $H(p,q) = \frac{p^2}{2m} + V(q)$, and taking the path integral measure to be

$$\mathcal{D}p\, \mathcal{D}q = \prod_{i=1}^{N} dp_i\, dq_i$$

show that this first–order path integral is equivalent to the usual path integral over $q(t)$ alone. Your answer should include a careful description of the resulting regularized measure $\mathcal{D}q$.

In real (Minkowskian) time, the wavefunction $\psi(q_f, T)$ for a non-relativistic particle of unit mass to be found at a location q_f at time T is given by the path integral

$$\psi(q_f;T) = \int \mathcal{D}q\ \exp\left(\mathrm{i}\int_0^T dt\left[\frac{1}{2}\dot{q}^2 - V(q)\right]\right) \times \psi(q_0, 0)$$

in terms of the initial wavefunction $\psi(q_0, 0)$. The path integral is taken over all intermediate values of the fields. Use the discretized path integral to show explicitly that $\psi(q_f;T)$ obeys the time–dependent Schrödinger equation. [*Hint: Take the measure for the field at the time step* i *to be* $dq_i/\sqrt{t_{i+1} - t_i}$ where $t_{N+1} \equiv T$.]

4 The gauge–fixed Lagrangian density for Yang-Mills theory is

$$\mathcal{L} = -\frac{1}{4g^2} F^{\mu\nu\, a} F_{\mu\nu}^{\ a} + (\partial^\mu b^a)\,(D_\mu c)^a + h^a\, \partial^\mu A_\mu^a - \frac{\xi}{2} h^a h^a$$

where the index a runs over a basis of the Lie algebra of the gauge group, h^a is the Nakanishi–Lautrup field and ξ is a constant gauge–fixing parameter.

a) Give expressions for the fieldstrength $F_{\mu\nu}^{\ a}$ and covariant derivative of the ghost field $(D_\mu c)^a$ in terms of ordinary derivatives, the gauge field A_μ^a and the structure constants of the Lie algebra of the gauge group.

b) The action above is invariant under the BRST transformations

$$\delta A_\mu^a = \epsilon\,(D_\mu c)^a\ , \qquad \delta c^a = -\frac{1}{2}\epsilon f^{a}_{\ bc} c^b c^c\,, \qquad \delta b^a = \epsilon h^a\,, \qquad \delta h^a = 0\,,$$

where ϵ is a constant anticommuting parameter. Obtain an expression for the corresponding conserved charge Q_{BRST}.

c) Assuming that the path integral measure is BRST invariant, derive the Ward identity

$$\langle [Q_{\mathrm{BRST}}, \mathcal{O}(x)] \cdots \rangle = 0$$

where $\mathcal{O}(x)$ is an arbitrary local operator, not necessarily BRST invariant, while the dots represent insertions of gauge invariant local operators built from $F_{\mu\nu}^{\ a}$ and its covariant derivatives, inserted away from x. What is the significance of this result?

END OF PAPER