MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 1:30 pm to 3:30 pm

PAPER 45

STATISTICAL FIELD THEORY

Attempt no more than ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Give an account of the features of a tricritical point and how it occurs in Landau–Ginsburg theory for a scalar-field order parameter M. Your account should include a clear discussion and explanation of the features of the 3D phase diagram containing a tricritical point.

Consider the case where the Landau–Ginsburg free energy is invariant under $M \to -M$. In the 2D phase diagram containing a tricritical point in which the x and y axes are the temperature T and the external field g respectively, the first-order line is parametrized by $T = T_0(g)$ and the second order line by $T = T_C(g)$. Explain how $T_0(g)$ and $T_C(g)$ are determined by the coefficients in the Landau–Ginsburg free energy and how the value of the critical temperature T_{TCP} for the tricritical phase transition is determined.

Near to a critical point the critical indices β and δ for this model are defined by

$$M \sim (-t)^{\beta} \ (t < 0, h = 0), \quad M \sim h^{1/\delta} \ (t = 0, h > 0) \ ,$$

where $t = (T - T_C)/T_C$, T_C is the critical temperature and h is the applied magnetic field. Calculate β and δ for a tricritical point.

The Blume-Capel model is defined on a square lattice Λ in D-dimensions where the lattice sites are labelled by the D-dimensional integer vector $\mathbf{n}$. The spin variable $\sigma_{\mathbf{n}}$ at the site $\mathbf{n}$ takes values $\sigma_{\mathbf{n}} \in [1, 0, -1]$. The Hamiltonian H is defined by

$$H \ = \ -J \sum_{\mathbf{n} \in \Lambda} \sigma_{\mathbf{n}} \sigma_{\mathbf{n}+\mu} \ + \ g \sum_{\mathbf{n} \in \Lambda} \sigma_{\mathbf{n}}^2 \, .$$

Use the mean field theory method to show that in this approximation the model exhibits tricritical behaviour with

$$T_{TCP} = \frac{2DJ}{3k_B} \ .$$

2

The Gaussian model in D-dimensions ($D < 4$) for a real scalar field ϕ is defined by the Hamiltonian

$$H(\phi) = \int_a d^D x \, \frac{1}{2}\Big(\kappa^{-1}(\nabla\phi(\mathbf{x}))^2 + m^2\phi(\mathbf{x})^2\Big) - J(\mathbf{x})\phi(\mathbf{x}) \ ,$$

where $\kappa > 0$ and a is the short-distance cut-off. Express H in terms of the Fourier transformed fields $\tilde{\phi}(\mathbf{p}), \tilde{J}(\mathbf{p})$. Derive explicitly an expression for the correlation length ξ in terms of the coupling constants.

Consider now the case where $J(\mathbf{x}) = h$, a constant. Construct a suitable Renormalization Group transformation for $H(\phi)$ by defining a blocking strategy with scale factor b and blocked field $\tilde{\phi}_B(\mathbf{p})$, followed by a rescaling of the momentum coordinates and the field such that

$$\mathbf{p} \longrightarrow \mathbf{q} = b\mathbf{p}, \qquad \tilde{\phi}_B(\mathbf{p}) \longrightarrow \tilde{\phi}(\mathbf{q}) : \ \tilde{\phi}_B(\mathbf{p}) = \tilde{Z}\tilde{\phi}(\mathbf{q}).$$

Show that $\tilde{Z}$ can be chosen such that under the Renormalization Group transformation

$$\kappa^{-1} \to (\kappa')^{-1} = \kappa^{-1}, \quad m \to m' = bm, \quad h \to h' = Z_h(b)h,$$

where you should explicitly determine $Z_h(b)$.

Assume that $m^2 \propto t$, where t is the temperature variable. Write down the relation between the free energies $F(t,h)$ and $F(t',h')$, defined per unit volume, of the original and rescaled Hamiltonians. Hence write the singular part of original free energy in the scaling form

$$F_S(t,h) = t^{2-\alpha} f(h/t^{\Delta}),$$

where you should identify the exponents α and Δ.

The Gaussian approximation in D-dimensions ($D < 4$) to the Hamiltonian for a Lifshitz point is

$$H(\phi) = \int_a d^{D-1}x_\perp \int_a dx_\parallel \, \frac{1}{2}\Big(\kappa^{-1}(\nabla_\perp\phi(\mathbf{x}))^2 + \mu^{-1}(\nabla_\parallel^2\phi(\mathbf{x}))^2 + m^2\phi(\mathbf{x})^2\Big) - h\phi(\mathbf{x}).$$

Here the D-dimensional position vector $\mathbf{x}$ is written as $\mathbf{x} = (\mathbf{x}_\perp, x_\parallel)$ where $\mathbf{x}_\perp$ is a $(D-1)$-dimensional Cartesian vector lying in the space $V_\perp$ orthogonal to the $D = 1$ space $V_\parallel$ that has coordinate $x_\parallel$. Define a suitable Renormalization Group transformation with scale factors b and c in the spaces $V_\perp$ and $V_\parallel$ respectively such that the momentum coordinates and the field rescale according to

$$\mathbf{p}_\perp \longrightarrow \mathbf{q}_\perp = b\mathbf{p}_\perp, \qquad \mathrm{p}_\parallel \longrightarrow \mathrm{q}_\parallel = c\mathrm{p}_\parallel, \quad \tilde{\phi}_B(\mathbf{p}) \longrightarrow \tilde{\phi}(\mathbf{q}) : \ \tilde{\phi}_B(\mathbf{p}) = \tilde{Z}\tilde{\phi}(\mathbf{q}).$$

Find c and $\tilde{Z}$ such that under the Renormalization Group transformation

$$\kappa^{-1} \to (\kappa')^{-1} = \kappa^{-1}, \quad \mu^{-1} \to (\mu')^{-1} = \mu^{-1}, \quad m \to m' = bm.$$

How does the field h rescale? Assuming that $m^2 \propto t$ identify the exponents α and Δ in this case.

3

A statistical system at temperature T is described by a scalar field theory whose effective Hamiltonian is given by

$$H = \int_{\Lambda^{-1}} d^D x \, \frac{1}{2}(\nabla\phi(\mathbf{x}))^2 + \frac{1}{2}m^2(\Lambda, T)\phi^2(\mathbf{x}) + \frac{1}{4!}g(\Lambda, T)\phi^4(\mathbf{x}) + \dots \; ,$$

where Λ is the ultra-violet cut-off. In this theory the two-point function $G(\mathbf{x})$ and its Fourier transform $\tilde{G}(\mathbf{p})$ are defined by

$$G(\mathbf{x}) \; = \; \langle \phi(0)\, \phi(\mathbf{x}) \rangle_c \; , \qquad \tilde{G}(\mathbf{p}) \; = \; \int d^D x \, e^{-i\mathbf{p}\cdot\mathbf{x}} \, G(\mathbf{x}) \; .$$

State what is meant by the truncated two-point function $\tilde{\Gamma}(\mathbf{p})$.

Using perturbation theory explain how $\tilde{\Gamma}(\mathbf{p})$ may be written as

$$\tilde{\Gamma}(\mathbf{p}) \; = \; \tilde{G}_0^{-1}(\mathbf{p}) + \delta m^2 + \Sigma(\mathbf{p}) \; ,$$

where the meaning of each of the terms in this expression should be clearly given. You may quote the rules of perturbation theory without derivation.

Hence show to one-loop order that

$$m^2(0, T) \; = \; m^2(\Lambda, T) \; + \; \frac{g}{2} \int^{\Lambda} \frac{d^D p}{(2\pi)^D} \, \frac{1}{\mathbf{p}^2 + m^2(0, T)} \; .$$

Show that this result is consistent with the Landau–Ginsburg assumption that $m^2(0, T) \sim (T - T_C)$ only for $D > D_C$, where the value of D_C for an ordinary critical point should be calculated.

Describe briefly how the value of D_C for a tricritical point is calculated and determine its value.

In the context of a scalar field theory, explain how the Ginsburg criterion can be used to show that the assumptions of Landau–Ginsburg theory are incorrect if the dimension $D \leqslant D_C$. Derive an expression for D_C in the case of a general critical point where the Landau–Ginsburg expression for the free energy in the neighbourhood of the critical point takes the form

$$A \; \sim \; -a_2 |t| M^2 + A_{2n} M^{2n} + \dots \; ,$$

where M is the scalar-field order parameter. Verify the results for D_C that you derived above.

END OF PAPER