UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 9:00 am to 12:00 pm

PAPER 44

SYMMETRIES, FIELDS AND PARTICLES

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let X, Y be elements of a matrix Lie algebra L. Prove the Baker–Campbell–Hausdorff (BCH) formula

$$\exp X \exp Y = \exp\left(X + Y + \frac{1}{2}[X, Y] + \dots\right)$$

up to the order shown. State, without proof, the next two terms in the expansion on the right hand side.

Let $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ denote the Pauli matrices and $\mathbf{1}$ the unit 2×2 matrix. Show that if α is real and $\mathbf{n}$ is a unit 3-vector, then

$$\exp(i\alpha\, \mathbf{n} \cdot \boldsymbol{\tau}) = \cos\alpha\, \mathbf{1} + i \sin\alpha\, \mathbf{n} \cdot \boldsymbol{\tau}\,,$$

and verify that this matrix is in $SU(2)$.

Evaluate $\exp i\alpha\tau_1 \exp i\beta\tau_2$ exactly, and show that the result is consistent with the BCH formula up to quadratic order in α, β.

2

Determine the Lie algebra of $SO(3)$. Show that the matrices

$$(T_a)_{bc} = -\epsilon_{abc} \qquad (a = 1, 2, 3)$$

form a basis for this Lie algebra, and find the structure constants.

A gauge theory with gauge group $SO(3)$ has a Higgs field Φ transforming under the fundamental vector representation. Write down the expressions for the covariant derivative $D_\mu\Phi$, and the field tensor $F_{\mu\nu}$. Give an example of a quartic Higgs potential $V(\Phi)$ and show that it is gauge invariant.

Given that the Higgs field has components $\{\Phi_c : c = 1, 2, 3\}$, and the gauge potential can be expressed as $A_\mu = A_{\mu a} T_a$, find expressions for $\{(D_\mu\Phi)_c : c = 1, 2, 3\}$ in component form.

Suppose now that $D_\mu\Phi = 0$ throughout spacetime. Show that with a suitable gauge choice, Φ has components $\{\Phi_1 = 0,\ \Phi_2 = 0,\ \Phi_3 = \phi\}$, where ϕ is constant. In this gauge, and assuming $\phi \neq 0$, find the general form of the gauge potential, and hence of the field tensor. Relate your results to the Higgs mechanism.

3

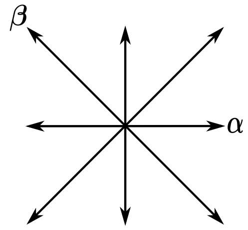
Define the group $SO(5)$ and show it has dimension 10.

Describe, explicitly, an $SO(4)$ subgroup of $SO(5)$. Show that the vector representation $\mathbf{5}$ and the adjoint representation $\mathbf{10}$ of $SO(5)$ decompose, respectively, into representations of $SO(4)$ as

$$\begin{aligned} \mathbf{5} &\rightarrow \mathbf{4} \oplus \mathbf{1}\,, \\ \mathbf{10} &\rightarrow \mathbf{6} \oplus \mathbf{4}\,. \end{aligned}$$

$SO(4)$ has a subgroup $SU(2)_{\mathrm{L}}$. What are the decompositions of the above $SO(4)$ representations into $SU(2)_{\mathrm{L}}$ irreducible representations?

The root diagram of $SO(5)$ is



where each pair of opposite roots is associated with an $SU(2)$ or $SO(3)$ subgroup, and $\boldsymbol{\alpha} = (1, 0)$, $\boldsymbol{\beta} = (-1, 1)$. $SO(5)$ weights $\boldsymbol{\lambda}$ need to satisfy $2\boldsymbol{\lambda} \cdot \boldsymbol{\alpha} \in \mathbb{Z}$, $\boldsymbol{\lambda} \cdot \boldsymbol{\beta} \in \mathbb{Z}$. Find the weight lattice and make a sketch of it incorporating the roots.

Show that $SU(2)_{\mathrm{L}}$ is associated with roots along a diagonal of the root diagram. Determine the weight diagrams of the $\mathbf{5}$ and $\mathbf{10}$ of $SO(5)$, giving your reasoning.

4

Plot diagrams of the flavour $SU(3)$ baryon and meson octets, using I_3 and Y eigenvalues as axes.

Define the orthonormal diagonal matrices h_1, h_2 of the Lie algebra of $SU(3)$. State, and justify, the formulae relating I_3 and Y to h_1 and h_2.

Using the electric charge assignments of the baryon octet, determine the electric charges of the quark triplet. Express the electric charge operator Q in terms of I_3 and Y, and also in terms of h_1 and h_2. Show that Q commutes with an $SU(2)$ subalgebra of the $SU(3)$ Lie algebra. What is the consequence of this for particles?

Briefly discuss the possible outcomes of collisions of pions (π^+, π^0, π^-) and nucleons (p, n), when the energy is just sufficient for the outgoing particles to be any meson-baryon pair from the meson and baryon octets. What conservation laws constrain the combinations of outgoing particles that can be observed?

END OF PAPER