UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 1:30 pm to 3:30 pm

PAPER 41

OPTIMAL INVESTMENT

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

An investor can invest in a single risky market, and in a bank account bearing interest at constant rate $r > 0$. He is required to pay a fraction $\tau \in (0,1)$ of his capital gains in tax, so that his wealth equation is

$$dw_t = rw_t\,dt + \theta\{\,\sigma dW_t + (\mu - r)\,dt\,\} - c_t dt - \tau d\bar{w}_t$$

where $\bar{w}_t = \sup_{s \leqslant t} w_s$. His objective is to achieve

$$V(w, \bar{w}) \equiv \sup_{(\theta,c)\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t} U(c_t)\, dt \,\middle|\, w_0 = w, \bar{w}_0 = \bar{w} \right]$$

where $U'(x) = x^{-R}$ for some $R > 0$ different from 1. Explain why V must have the scaling property

$$V(w, \bar{w}) = \bar{w}^{1-R}\, v(w/\bar{w}).$$

Derive the HJB equation for this problem. In terms of the dual variable $z = v'(x)$, derive an ODE satisfied by the dual value function $J(z) = v(x) - xz$, and hence prove that for some constants A and z_*

$$J(z) = -\frac{\tilde{U}(z)}{Q(1 - 1/R)} + A\left(\frac{z}{z_*}\right)^{-\alpha},$$

where $-\alpha < 0$ is the negative root of the quadratic

$$Q(t) = \tfrac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho.$$

Assuming that $\gamma_M \equiv -Q(1 - 1/R) > 0$, prove that

$$\begin{aligned} z_* &= \left\{ \frac{\alpha R + R - 1}{\gamma_M(\alpha R + R - 1 + \alpha\tau)} \right\}^R, \\ A &= -\frac{\tau\, z_*}{\alpha R + R - 1}. \end{aligned}$$

2

(i) The assets under management of a hedge fund at the start of a year ($t = 0$) are w_0, and evolve up to the end of the year ($t = T$) as

$$dw_t = rw_t\, dt + \theta_t\{\, \sigma dW_t + (\mu - r)dt\,\}.$$

Here, the portfolio process θ is the amount of the fund's assets which the manager chooses to invest in the single risky asset, W is a standard Brownian motion, and r, σ and μ are constants. Define the *state-price density* process ζ for this problem. Assuming that u is C^2, increasing, strictly concave, and satisfies the Inada conditions, characterize the optimal solution w_T^* to

$$\sup_{\theta} Eu(w_T) \qquad \text{subject to } w_0 = w$$

in terms of the state-price density. Assuming that the w_T^* which you propose satisfies the budget constraint, prove that it is optimal.

(ii) At the end of the year ($t = T$), the manager's contract rewards him with

$$X = \varepsilon w_T + \alpha(w_T - w_0)^+,$$

where α and ε are positive constants. The manager has CARA utility $U(x) = -\exp(-\gamma x)$, and chooses θ so as to obtain

$$\sup_{\theta \in \mathcal{A}} EU(X),$$

where $\mathcal{A}$ is the set of admissible portfolio processes θ which keep the fund's wealth always non-negative. Show that the map

$$w_T \mapsto U(X)$$

is increasing but not concave. Assuming that w_0 is large enough, explain why the fund manager's problem is equivalent to the problem

$$\sup_{\theta \in \mathcal{A}} E\bar{U}(w_T)$$

for some concave increasing function $\bar{U}$ related to U and the constants α, ε in a way that you should specify. Explain how the optimal solution for the manager is expressed in terms of the state-price density.

In the special case where $r = \mu = 0$, describe briefly how the fund manager will act.

3

Let S^i_t denote the price at time t of stock i, $i = 1, \ldots, n$, and let $S^0_t \equiv M_t$ denote the value at time t of some market index. The dynamics of the stocks are given by

$$dS^i_t/S^i_t = \mu_i \, dt + \sigma_i \{ \rho(M_t) \, dW^0_t + \tilde{\rho}(M_t) \, dW^i_t \} \qquad (i = 1, \ldots, n)$$

where $\rho : (0, \infty) \to (-1, 1)$ is smooth and decreasing, and $\tilde{\rho}(m) \equiv \sqrt{1 - \rho(m)^2}$. The W^i are independent standard Brownian motions, and μ_i, σ_i are positive constants. The index evolves as

$$dM_t/M_t = \mu_0 \, dt + \sigma_0 \, dW^0_t.$$

An investor is able to invest in the n stocks, the market index, and in a riskless bank account bearing constant interest rate r. He consumes over time in such a way as to achieve his objective

$$\sup_{(\theta,c) \in \mathcal{A}} E \int_0^\infty e^{-\beta t} U(c_t) \, dt$$

where $\beta > 0$ is constant, U is a utility with constant coefficient $R > 0$ of relative risk aversion, and $\mathcal{A}$ is the set of admissible portfolio-consumption pairs.

Write down the wealth equation for this investor, and derive the HJB equation, taking care to exploit and explain any scaling relations that simplify the equations. Develop the HJB equation to the point where it is a second-order non-linear ODE for an unknown function f of the variable $m \equiv \log M$. Obtain expressions for the optimal θ and c in terms of f. Briefly discuss how this problem could be solved numerically.

4

An investor can invest in a bank account bearing interest at constant rate $r > 0$ and in a single risky asset, and he can consume in such a way that his wealth remains always non-negative. His wealth evolves as

$$dw_t = rw_t\,dt + \theta_t(\sigma dW_t + (\mu - r)\,dt) - c_t\,dt$$

where W is a standard Brownian motion, and μ and σ are constants. His objective is to achieve

$$V(w, \bar{c}) \equiv \sup_{(\theta,c)\in\mathcal{A}} E\left[\int_0^\infty e^{-\rho t} g(c_t, \bar{c}_t)\,dt \,\middle|\, w_0 = 0, \bar{c}_0 = \bar{c} \right],$$

where for some $\lambda > 0$

$$\bar{c}_t = \int_0^t \lambda\, e^{\lambda(s-t)}\, c_s\, ds + e^{-\lambda t}\, \bar{c}_0,$$

and

$$g(c, \bar{c}) = U(c)\,(c/\bar{c})^{-\alpha}$$

where $\alpha > 0$ and $U(x) = x^{1-R}/(1-R)$ for some $R > 1$.

Derive the differential equation satisfied by $\bar{c}$. Explain why the value function V must scale as

$$V(w, \bar{c}) = \bar{c}^{1-R}\, v(w/\bar{c}).$$

Derive the HJB equation satisfied by V, and rework this into an ODE for v. Find the dual form of this equation, and comment on the form it takes when $\lambda = 0$.

END OF PAPER