UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 1:30 pm to 4:30 pm

PAPER 40

ADVANCED FINANCIAL MODELS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a two asset market model whose prices have dynamics

$$\begin{aligned} dB_t &= B_t r \; dt \\ dS_t &= S_t(\mu \; dt + \sigma \; dW_t) \end{aligned}$$

where r, μ, σ are positive constants and W is a Brownian motion. Consider a perpetual American claim with payout $g(S_t)$, where g is a given non-negative function. Suppose there is a non-negative function V which satisfies the non-linear differential equation

$$\max\left\{\frac{1}{2}\sigma^2 S^2 V''(S) + rSV'(S) - rV(S), g(S) - V(S)\right\} = 0 \text{ for } S > 0. \qquad (*)$$

(a) Show that if the initial wealth is at least $V(S_0)$, then there is an admissible self-financing strategy (ϕ, π) such that $\phi_t B_t + \pi_t S_t \geqslant g(S_t)$ almost surely for all $t \geqslant 0$.

(b) Now suppose that $2r/\sigma^2 < 1$ and $g(S) = \frac{1}{1+S}$ for all $S > 0$. Find, with verification, a solution to equation $(*)$ and hence an optimal exercise policy of this claim.

2

Let p be a given vector in $\mathbb{R}^n$, and let P be a bounded $\mathbb{R}^n$-valued random vector. Suppose that the following condition holds: if $H \in \mathbb{R}^n$ is such that $H \cdot p = 0 = H \cdot P$ almost surely, then $H = 0$.

(a) Show that the following statements are equivalent:

1. There exists a positive bounded random variable Y such that $\mathbb{E}(PY) = p$.
2. If $H \in \mathbb{R}^n$ is such that $H \cdot p \leqslant 0 \leqslant H \cdot P$ almost surely, then $H = 0$.

If you use any result from the course, you must prove it.

Now suppose that the equivalent statements (1) and (2) hold. Also suppose $n = 3$ and that $p = (b, s, c)$ and $P = (B, S, C)$ where B and S are bounded, non-negative random variables and $C = (S - B)^+$.

(b) Show that

$$(s - b)^+ \leqslant c \leqslant s.$$

Furthermore, show that if $0 < \mathbb{P}(S > B) \leqslant \mathbb{P}(S \geqslant B) < 1$ then $(s - b)^+ < c$.

3

Consider a discrete-time, arbitrage-free, n-asset market model with prices $(P_t)_{t \geqslant 0}$ adapted to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$, where $\mathcal{F}_0$ is trivial.

(a) What does it mean to say the market is complete?

Suppose that the market is complete.

(b) Fix $t \geqslant 0$ and let $A_1, \ldots, A_k$ be disjoint $\mathcal{F}_t$-measurable events of positive probability. Show that $k \leqslant n^t$.

(c) Let $V_t = \mathbb{E}(P_t P_t^\top | \mathcal{F}_{t-1})$ where $P_t^\top$ denotes the transpose of the column vector P_t. Suppose that the $n \times n$ random matrix V_t is positive definite almost surely for each $t \geqslant 1$, and let

$$Z_t = P_t^\top V_t^{-1} P_{t-1}.$$

Show that $Z_t > 0$ almost surely for each $t \geqslant 1$. [Hint: You may wish to show that the process $Y_t = Z_1 \cdots Z_t$ defines a martingale deflator. You may use without proof the first fundamental theorem of asset pricing if stated clearly.]

4

Consider a two-asset market with cash $B_t = 1$ for all $t \geqslant 0$ and a stock with price dynamics

$$dS_t = S_t \sigma_t dW_t$$

where $(W_t)_{t \geqslant 0}$ is a Brownian motion and $(\sigma_t)_{t \geqslant 0}$ is a bounded and continuous process adapted to filtration $(\mathcal{F}_t)_{t \geqslant 0}$.

(a) Show that the process $(\sqrt{S_t})_{t \geqslant 0}$ is a supermartingale.

(b) If the process σ is independent of W, show that

$$\mathbb{E}(\sqrt{S_T} | \mathcal{F}_t) = \sqrt{S_t} e^{-k \int_t^T \sigma_u^2 du}$$

for a constant k which you should determine.

Let $(f_t(T))_{0 \leqslant t \leqslant T}$ be a continuous random field which evolves as

$$df_t(T) = A_t(T) dt + B_t(T) dW_t$$

where $(A_t(T))_{0 \leqslant t \leqslant T}$ and $(B_t(T))_{0 \leqslant t \leqslant T}$ are bounded, continuous and suitably measurable random fields such that the stochastic integral is well-defined for all $T \geqslant 0$.

(c) If

$$\mathbb{E}(\sqrt{S_T} | \mathcal{F}_t) = \sqrt{S_t} e^{-\int_t^T f_t(u) du}$$

for all $0 \leqslant t \leqslant T$, show that

$$A_t(T) = B_t(T) \left(\int_t^T B_t(u) du - \frac{1}{2} \sigma_t \right)$$

and

$$f_t(t) = k \ \sigma_t^2 .$$

[You may use the stochastic Fubini theorem without justification.]

5

Consider a two-asset discrete-time market with a bond with price $B_t = 1$ for all $t \geqslant 0$ and a stock with price $(S_t)_{t\geqslant 0}$.

(a) Consider a contingent claim with time-T payout

$$\sum_{u=1}^{T} S_u.$$

Find a trading strategy that replicates this claim.

Now suppose the random variable S_T takes values in the finite set $\{0, 1, \ldots, N\}$. Suppose the market has a family of N European call options each with maturity T and with strikes $K \in \{1, \ldots, N\}$.

(b) Show that the claim with time-T payout S_T^2 can be replicated by trading in the stock and the family of call options.

(c) Consider a contingent claim with time-T payout

$$\sum_{t=1}^{T} (S_t - S_{t-1})^2.$$

Show that the claim can be replicated by trading in the bond, the stock and the family of call options. Show that the initial cost of the replication strategy is

$$2 \sum_{K=1}^{N} C_0(K) + S_0(1 - S_0)$$

where $C_0(K)$ is the time-0 price of the call of strike K.

6

Consider a two asset market model with price dynamics

$$\begin{aligned} dB_t &= B_t r \ dt \\ dS_t &= S_t(r \ dt + \sigma(t, S_t) dW_t) \end{aligned}$$

where r is a positive constant, the process W is a Brownian motion generating the filtration, and the positive, smooth function σ is bounded from above and below. To this market, introduce a European call option with strike K and maturity T. Let

$$\mathbb{E}[e^{-rT}(S_T - K)^+] = C(T, K).$$

(a) Show that there exists a self-financing admissible strategy which replicates the payout of the call. Show that the minimal cost of replication is $C(T, K)$. [You may use without proof standard results of stochastic calculus if stated clearly.]

(b) Derive Dupire's PDE for $C(T, K)$. [You may use without proof the following fact from stochastic calculus: for all $t > 0$, the random variable S_t has a density $\psi(t, \cdot)$. Furthermore, the function ψ is continuous and satisfies the equation

$$\mathbb{E}[(S_T - K)^+] = (S_0 - K)^+ + \int_0^T \int_K^\infty rs\psi(t, s) ds \ dt + \frac{1}{2} \int_0^T K^2 \sigma(t, K)^2 \psi(t, K) dt$$

for all $K, T > 0$.]

(c) Let $P(T, K)$ be the minimal replication cost of a European put with strike K and maturity T. Show that $P(T, K)$ also satisfies Dupire's PDE.

END OF PAPER