UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 9:00 am to 11:00 am

# PAPER 4

# HOMOLOGICAL AND HOMOTOPICAL ALGEBRA

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

Let $R$ be an arbitrary unital ring.

Let $\mathcal{E}(C, A)$ be the set of short exact sequences $0 \to A \to B \to C \to 0$ of left $R$-modules up to equivalence. Prove that there is a bijective correspondence between $\mathcal{E}(C, A)$ and $\mathrm{Ext}^1_R(C, A)$.

Let $\mathbb{Z}$ be the trivial module for the abelian group $\mathbb{Z}$. Classify, up to equivalence, all short exact sequences of the form

$$0 \to \mathbb{Z} \to B \to \mathbb{Z} \oplus \mathbb{Z} \to 0.$$

**2**

Let $R$ be an arbitrary unital ring. Define the Koszul complex $K(\mathbf{t})$ of a sequence $\mathbf{t} = (t_1, \ldots, t_n)$ of central elements of $R$.

Let $M$ be a left $R$-module. Define what it means for a sequence to be regular on $M$. Show that if $\mathbf{t}$ is regular on $M$ then $K(\mathbf{t}) \otimes_R M$ is quasi-isomorphic to $M/(t_1, \ldots, t_n)M$. You should state carefully any results that you use.

Consider the ring $\mathbb{Z}[X]$ and its modules $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/q\mathbb{Z}$ on which $X$ acts trivially. (Here $p, q \geqslant 2$.) Compute $\mathrm{Ext}^*_{\mathbb{Z}[X]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/q\mathbb{Z})$ in the cases where (i) $p = q$ and (ii) $p$ and $q$ are coprime.

**3** Define what it means to say a spectral sequence converges to a graded object $H^*$.

Let $R$ be an arbitrary unital ring, let $C$ be a cochain complex of left $R$-modules and let $F$ be a bounded filtration on $C$. State a theorem about the convergence of the spectral sequence associated to the filtered complex $(C, F)$.

Describe how to deduce that there are two spectral sequences computing the cohomology of a double complex that is bounded in both degrees.

Let $P$ be a bounded cochain complex of projective left $R$-modules and let $M$ be any left $R$-module with finite projective dimension. By considering the two spectral sequences associated to a suitable double complex prove the following Künneth spectral sequence:

$$E_2^{pq} = \mathrm{Tor}_{-p}(H^q P, M) \Rightarrow H^{p+q}(P \otimes M)$$

Assume now that all boundaries of the complex $P_*$ are also projective. Show the spectral sequence degenerates at the $E_2$ term.

**4**

Define the the weak equivalences, fibrations and cofibrations that form the projective model structure on the category $\mathbf{Ch}_{\geqslant 0}(R)$ of nonnegative chain complexes over an arbitrary unital ring $R$.

Define what is meant by a sequentially small object. Use the small object argument to show that any map $a : X \longrightarrow Y$ in this category factors as a cofibration followed by an acyclic fibration.

*Hint:* You may assume that acyclic fibrations are characterized by the right lifting property with respect to a certain set of cofibrations that you should specify. You may further assume the domains of these maps are sequentially small.

**END OF PAPER**