UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 1:30 pm to 4:30 pm

PAPER 39

STOCHASTIC NETWORKS

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Prove that in equilibrium the departure process from an M/M/K queue is a Poisson process.

Airline passengers arrive at passport control in accordance with a Poisson process of rate ν. The control operates as a two server queue at which service times are independent and exponentially distributed with mean $\mu < 2/\nu$ and are independent of the arrival process. After leaving passport control, a passenger must pass through a security check. This operates as a single-server queue at which service times are all equal to a constant $\tau(< \nu^{-1})$. Show that in equilibrium the probability both queues are empty is

$$\frac{2-\nu\mu}{2+\nu\mu}\,(1-\nu\tau)\,.$$

If it takes a time σ to walk from the first queue to the second, what is the equilibrium probability that both queues are empty and there is no passenger walking between them?

2

Derive Erlang's formula, $E(\nu, C)$, for the proportion of calls lost at a resource of capacity C offered a load of ν. State clearly any assumptions you make in your derivation.

Define a loss network with a fixed routing, and describe briefly how the Erlang fixed point equations

$$B_j = E\left((1-B_j)^{-1}\sum_r A_{jr}\nu_r \prod_i (1-B_i)^{A_{ir}}, C_j\right), \quad j = 1, 2, \ldots J \tag{1}$$

arise as a natural approximation for the link blocking probabilities in the network, where A is the link-route incidence matrix, assumed to be a $0-1$ matrix.

Show that a solution to the equations (1) also locates the minimum of a strictly convex function

$$\sum_r \nu_r e^{-\sum_j y_j A_{jr}} + \sum_j \int_0^{y_j} U(z, C_j)dz$$

over the positive orthant $y \geqslant 0$, for a function U to be determined.

Describe a form of repeated substitution that converges to a solution of equations (1), and prove that it converges.

3

Define a *Wardrop equilibrium* for the flows $x = (x_r, r \in R)$ in a congested network with routes $r \in R$ and links $j \in J$.

Show that if the delay $D_j(y_j)$ at link j is a continuously differentiable, strictly increasing function of the throughput, y_j, of the link j then a Wardrop equilibrium exists and solves an optimization problem of the form

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in J} \int_0^{y_j} D_j(u)du \\ \text{over} \quad & x \geqslant 0, \quad y \\ \text{subject to} \quad & Hx = f, \qquad Ax = y \ , \end{aligned}$$

where $f = (f_s, s \in S)$ and f_s is the (fixed) aggregate flow between source-sink pair s. What are the matrices A and H? Are the equilibrium throughputs, y_j, unique? Are the equilibrium flows, x_r, unique? Justify your answers.

Suppose now that, in addition to the delay $D_j(y_j)$, users of link j incur a traffic-dependent toll $T_j(y_j)$, and suppose each user selects a route in an attempt to minimize the sum of its toll and its delays. Show that it is possible to choose the functions $T_j(\cdot)$ so that the equilibrium flow pattern minimizes the average delay in the network.

4

The dynamical system

$$\begin{aligned} \frac{d}{dt}\mu_j(t) &= \kappa_j \mu_j(t) \left(\sum_r A_{jr} x_r(t) - C_j \right) & j \in J \\ x_r(t) &= \frac{w_r}{\sum_k \mu_k(t) A_{kr}} & r \in R \end{aligned}$$

is proposed as a model for resource allocation in a network, where R is a set of routes, J is a set of resources, A is a $0-1$ matrix, $w_r > 0$ for $r \in R$, C_j is the capacity of resource j and $\kappa_j > 0$ for $j \in J$.

Provide a brief interpretation of this model as a process that attempts to balance supply and demand.

By considering the function

$$V(\mu) = \sum_{r \in R} w_r \log \left(\sum_j \mu_j A_{jr} \right) - \sum_{j \in J} \mu_j C_j$$

or otherwise, show that if the matrix A has full rank then all trajectories of the dynamical system converge toward a unique equilibrium point. What happens if A is not of full rank?

5

Let J be a set of resources, and R a set of routes, where a route $r \in R$ identifies a subset of J. Let C_j be the capacity of resource j, and suppose the number of flows in progress on each route is given by the vector $n = (n_r, r \in R)$. Define a proportionally fair rate allocation.

Consider a network with resources $J = \{1, 2\}$, each of unit capacity, and routes $R = \{\{1\}, \{2\}, \{1, 2\}\}$. Given $n = (n_r, r \in R)$, find the rate x_r of each flow on route r, for each $r \in R$, under a proportionally fair rate allocation.

Suppose now that flows describe the transfer of documents through a network, that new flows originate as independent Poisson processes of rates ν_r, $r \in R$, and that document sizes are independent and exponentially distributed with parameter μ_r for each route $r \in R$. Determine the transition intensities of the resulting Markov process $n = (n_r, r \in R)$. Show that the stationary distribution of the Markov process $n = (n_r, r \in R)$ takes the form

$$\pi(n) = B \binom{n_{\{1\}} + n_{\{2\}} + n_{\{1,2\}}}{n_{\{1,2\}}} \prod_{r \in R} \left(\frac{\nu_r}{\mu_r} \right)^{n_r},$$

where B is a normalizing constant, provided the parameters $(\nu_r, \mu_r, r \in R)$ satisfy certain conditions. Determine these conditions, and also the constant B.

Show that, under the distribution π, $n_{\{1\}}$ and $n_{\{2\}}$ are independent. Are $n_{\{1\}}$, $n_{\{2\}}$ and $n_{\{1,2\}}$ independent?

END OF PAPER