UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 9:00 am to 12:00 pm

PAPER 38

MATHEMATICS OF OPERATIONAL RESEARCH

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State and prove the Lagrangian sufficiency theorem.

Consider the problem to maximize or minimize $3y - z$ subject to $2x - y - z \leqslant 2$ and $x^2 + y^2 = 5$, where $x, y, z \in \mathbb{R}$.

(b) Use Lagrangian sufficiency and complementary slackness to find an optimal solution for the maximization problem.

(c) Explain why the same method does not yield an optimal solution for the minimization problem, and show that the minimization problem is in fact unbounded.

(d) Find an optimal solution for the minimization problem subject to the additional constraint that $z \leqslant 0$.

2

Let $P = \{x \in \mathbb{R}^n : Ax \leqslant b\}$ and $Q = \{y \in \mathbb{R}^n : Cy \leqslant d\}$, where $A, C \in \mathbb{R}^{m \times n}$ and $b, d \in \mathbb{R}^m$ such that $P \neq \emptyset$ and $Q \neq \emptyset$.

(a) Using the simplex method or otherwise, find $x \in P \cap Q$ when

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -6 & -4 \\ 0 & 0 & -1 \end{pmatrix}, \qquad b = \begin{pmatrix} 7 \\ -8 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \qquad d = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}.$$

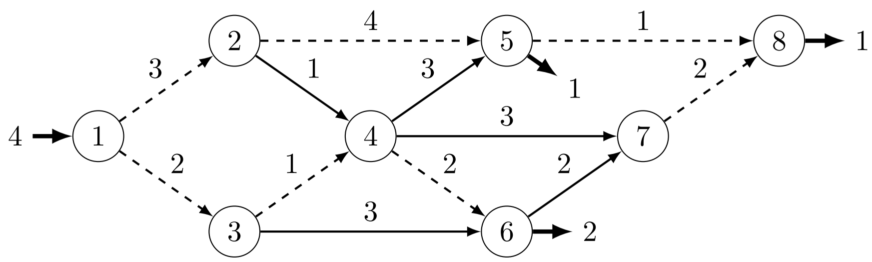
(b) Show that $P \cap Q = \emptyset$ if and only if there exists $h \in \mathbb{R}^n$ such that for all $x \in P$ and $y \in Q$, $h^T x < h^T y$. The implication in one direction is straightforward. For the other direction, use Lagrangian duality to show that $P \cap Q = \emptyset$ implies that there exist $\lambda^T, \mu^T \geqslant 0$ such that $\lambda^T A + \mu^T C = 0$ and $\lambda^T b + \mu^T d < 0$.

3

Consider the uncapacitated minimum cost flow problem on a network $G = (V, E)$.

(a) Define the problem formally. Derive the Lagrangian, as well as dual feasibility and complementary slackness conditions that can be used to identify an optimal solution. Explain how the optimality conditions are related to spanning trees of G.

Now consider the following network with four units of flow entering at vertex 1, two units leaving at vertex 6, and one unit each leaving at vertices 5 and 8. Each edge (i, j) is labeled with the cost c_{ij} of sending one unit of flow along it. The dashed edges indicate a spanning tree T.



(b) Find the basic feasible solution corresponding to T. Starting from this solution, carefully explain the network simplex method and show that it finds an optimal solution in one step.

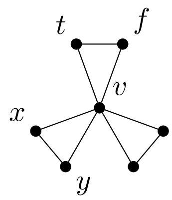
4

A k-coloring of an undirected graph (V, E) is a function $c : V \to \{1, \ldots, k\}$ such that for every $(u, v) \in E$, $c(u) \neq c(v)$. A graph is called k-colorable if it has a k-coloring.

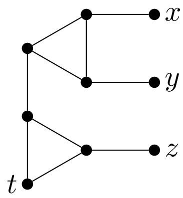
(a) Define the complexity classes P and NP-complete.

(b) Show that deciding 2-colorability of a graph is in P.

(c) Show that for any 3-coloring c of the following graph, $|\{c(t), c(f)\}| = |\{c(x), c(y)\}| = |\{c(t), c(f), c(x), c(y)\}| = 2$:



(d) Show that for a 3-coloring c of the following graph, it cannot be the case that $c(x) = c(y) = c(z) \neq c(t)$:



(e) Show that the problem of deciding whether a graph is 3-colorable is NP-complete. You may assume NP-completeness of the satisfiability problem for Boolean formulae in conjunctive normal form with at most three literals per clause.

5

Call $A \in \mathbb{R}^{n \times n}$ *doubly stochastic* if $A \geqslant 0$ and $\sum_{i=1}^{n} a_{ij} = \sum_{i=1}^{n} a_{ji} = 1$ for all $j \in \{1, \ldots, n\}$. Call a matrix $P \in \mathbb{R}^{n \times n}$ a *permutation matrix* if it is doubly stochastic and $P \in \{0,1\}^{n \times n}$.

(a) Let $A \in \mathbb{R}^{n \times n}$ be doubly stochastic and let $B \in \mathbb{R}^{n \times n}$ such that for all $i, j \in \{1, \ldots, n\}$,

$$b_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Use Hall's theorem to show that there exists a permutation matrix P and a matrix $C \geqslant 0$ such that $B = P + C$.

(b) Show that any doubly stochastic matrix A can be written as a convex combination of permutation matrices, i.e., that $A = \sum_{i=1}^{k} \alpha_i P_i$, where $P_1, \ldots, P_k \in \mathbb{R}^{n \times n}$ are permutation matrices and $\alpha \in \mathbb{R}^k$ with $\alpha \geqslant 0$ and $\sum_{i=1}^{k} \alpha_i = 1$.

(c) Provide upper bounds on k and on the running time of an algorithm that determines α_i and P_i for all $i \in \{1, \ldots, k\}$. Explain briefly why these upper bounds are correct.

6

Recall that the Lemke-Howson algorithm finds a Nash equilibrium of a non-degenerate bimatrix game by complementary pivoting.

(a) Write down the linear inequalities maintained by the algorithm in terms of the payoff matrices $P, Q \in \mathbb{R}^{m \times n}$ and vectors x and y corresponding to the strategies of the two players. Give the termination condition and show that this condition yields an equilibrium.

Consider now the following pair of tableaus, which have been obtained by applying the Lemke-Howson algorithm to a particular non-degenerate bimatrix game:

x_1	x_2	x_3	s_4	s_5	s_6	
0	1	0	$\frac{1}{3}$	0	$-\frac{1}{9}$	$\frac{2}{9}$
0	0	3	0	1	$-\frac{1}{3}$	$\frac{2}{3}$
1	0	1	$-\frac{1}{3}$	0	$\frac{4}{9}$	$\frac{1}{9}$

y_4	y_5	y_6	r_1	r_2	r_3	
$\frac{3}{4}$	$\frac{15}{4}$	0	1	$-\frac{1}{4}$	0	$\frac{3}{4}$
$\frac{1}{4}$	$\frac{1}{4}$	1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$\frac{9}{4}$	$\frac{9}{4}$	0	0	$-\frac{3}{4}$	1	$\frac{1}{4}$

(b) Find an equilibrium (x', y') of the game.

(c) Determine the payoff matrices of the game up to affine transformation, and show that (y', x') is also an equilibrium.

END OF PAPER