UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 1:30 pm to 4:30 pm

PAPER 37

TIME SERIES AND MONTE CARLO INFERENCE

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider the process $\{X_t, t \in \mathbb{Z}\}$:

$$X_t = \phi X_{t-1} + \epsilon_t, \qquad \text{with } \epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2).$$

(a) Carefully explain whether $\{X_t\}$ can be weakly stationary in the following scenarios:

(i) $\phi = 1/2$;

(ii) $\phi = 1$;

(iii) $\phi = 2$.

(b) What is the motivation for testing $\phi = 1$? Given observations $X_1, \ldots, X_n$, how would you perform such a test of size α? Write down the test statistic and the critical region.

[Derivation of the test statistic's asymptotic distribution is not required here.]

Now consider an autoregressive (AR) process $\{Y_t, t \in \mathbb{Z}\}$ of order p, where

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \epsilon_t, \qquad \text{with } \epsilon_t \overset{i.i.d.}{\sim} N(0, \sigma^2).$$

(c) What does it mean to say $\{Y_t\}$ is "causal"? Prove that if $\{Y_t\}$ is causal, then the roots of the equation

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = 0$$

all lie outside the unit circle.

(d) Suppose $\{Y_t\}$ is causal. Show that its autocovariance function γ_h decays exponentially, i.e., there exist constant $C > 0$ and $s \in (0, 1)$ such that $|\gamma_h| \leqslant Cs^{|h|}$ for $h = 0, \pm 1, \pm 2, \ldots$.

2

Consider the following autoregressive conditional heteroscedasticity (ARCH) process $\{X_t, t \in \mathbb{Z}\}$:

$$X_t = \sigma_t \epsilon_t, \quad \epsilon_t \overset{i.i.d.}{\sim} N(0,1),$$
$$\sigma_t^2 = \alpha_0 + \alpha_2 X_{t-2}^2,$$

with $\alpha_0 > 0$ and $1 > \alpha_2 > 0$. You may assume that under these conditions $\{X_t\}$ is both weakly stationary and strongly stationary.

(a) What is meant by "weakly stationary" and "strongly stationary"?

(b) List two reasons why researchers often prefer ARCH processes to autoregressive moving average (ARMA) processes when modelling financial time series.

(c) Calculate EX_t, EX_t^2, $\mathrm{cov}(X_t^2, X_{t+1}^2)$ and EX_t^4. Deduce that $EX_t^4 < \infty$ if $3\alpha_2^2 < 1$.

[You can directly use the fact that the fourth moment of a standard normal random variable is 3.]

(d) Write down the autoregressive representation of this ARCH process. Now given observations $X_1, \ldots, X_n$, carefully explain how you would estimate α_0 and α_2 using the least squares method.

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be any function with $E[f^2(X_t)] < \infty$ and h be any positive integer. Prove or disprove the following statements:

(i) $\mathrm{cov}(X_{t+h}, f(X_t)) = 0$;

(ii) $\mathrm{cov}(X_t, f(X_{t+h})) = 0$.

3

(a) Define the autocorrelation function (ACF) and partial autocorrelation function (PACF) of a zero-mean weakly stationary process.

(b) Briefly describe how the sample ACF and PACF can be used to obtain a good guess of appropriate orders in autoregressive moving average (ARMA) models given a particular time series data set.

Now consider the following invertible moving average (MA) process $\{X_t, t \in \mathbb{Z}\}$ of order q:

$$X_t = \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}, \qquad \text{with } \epsilon_t \stackrel{i.i.d.}{\sim} N(0, \sigma^2).$$

(c) If $q = 1$, compute the PACF of this MA(1) process at lag 2.

(d) Now assume $q \in \{1, 2, 3, \ldots\}$. Use the spectral representation theorem to prove that the spectral density of $\{X_t\}$ is $f_X(\omega) = \sigma^2 |\Theta(e^{2\pi i \omega})|^2$, where $\Theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q$.

(e) Prove that the variance of $\left(\sum_{t=1}^{n} X_t\right)/\sqrt{n}$ converges to $\sigma^2 \Theta^2(1)$ as $n \to \infty$. How about the variance of $\left(\sum_{t=1}^{n} X_{2t-1}\right)/\sqrt{n}$?

4

(a) Let $F : \mathbb{R} \to [0,1]$ be a CDF. Define its associated quantile function $F^{-1} : (0,1) \to \mathbb{R}$ and prove that $F^{-1}(U)$ has CDF F if $U \sim \text{Unif}[0,1]$.

(b) Let $f(x)$ be a density function on $\mathbb{R}^d$ and $h : \mathbb{R}^d \to \mathbb{R}$ be a function such that $\int_{\mathbb{R}^d} |h(x)|f(x)dx < \infty$. Define the importance sampling estimator $\hat{\mu}_{\text{IS}}$ of the integral $\mu = \int_{\mathbb{R}^d} h(x)f(x)dx$, and show that it is unbiased. Give the importance distribution that minimizes the variance of $\hat{\mu}_{\text{IS}}$, and prove your claim.

(c) (i) Given $U_1 \sim \text{Unif}[0,1]$, how can you simulate $X \sim f$, where $f(x) = \frac{\alpha}{\beta}x^{\alpha-1}\exp(-x^\alpha/\beta)$ if $x > 0$, $f(x) = 0$ if $x \leqslant 0$, and $\alpha, \beta > 0$.

(ii) Given $U_2, U_3 \overset{\text{iid}}{\sim} \text{Unif}[0,1]$, give the Box-Muller algorithm to generate $X_1, X_2 \overset{\text{iid}}{\sim} N(0,1)$, and prove that it works.

(iii) Given $U_1, U_2, U_3 \overset{\text{iid}}{\sim} \text{Unif}[0,1]$, explain how can you simulate from the density function $g(x)$ defined by

$$g(x) = (2\pi)^{-1/2} \int_0^\infty \exp\left(-\frac{x^2 + \theta^4}{2\theta^2}\right) d\theta, \quad x \in \mathbb{R}.$$

5

(a) Define the term "φ-irreducibility" for a Markov Chain on a state space $S \subset \mathbb{R}^d$, and explain what it intuitively means.

(b) State the Ergodic Theorem and the Central Limit Theorem (CLT) for Markov Chains on a state space $S \subset \mathbb{R}^d$. (please explain the different terms involved in the asymptotic variance of the CLT).

(c) Let $\pi : \mathbb{R}^d \to [0, \infty)$ be an unnormalized density, from which we wish to sample. State sufficient conditions under which the general Metropolis-Hasting's algorithm with proposal density $q(y|x)$ and stationary distribution $\pi(x)/\int_{\mathbb{R}^p} \pi(y)dy$ satisfies the assumptions of the Central Limit Theorem for Markov Chains (no proof is needed).

(d) Let $\pi(\cdot)$ be an unnormalized density on $\mathbb{R}^p$, and denote by

$$\pi_i(\cdot|x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p)$$

the full conditional density of $X_i|(X_j : j \neq i)$, where

$$X = (X_1, \ldots, X_p) \;\sim\; \pi.$$

Consider the following algorithm:

I. Choose $x_0 \in \mathbb{R}^p$ such that $\pi(x_0) > 0$, and set $X^{(0)} = x_0$.

II. For each $t = 0, 1, \ldots, T-1$,

1) choose $k_t \in \{1, \ldots, p\}$ uniformly at random,
2) draw $Y_t \sim \pi_{k_t}\left(\cdot\middle|X_j^{(t)} : j \neq k_t\right)$,
3) set

$$X_j^{(t+1)} = \begin{cases} X_j^{(t)}, & j \neq k_t, \\ Y_t, & j = k_t. \end{cases}$$

(i) Give the transition kernel $p(x, x')$ associated to this Markov Chain.

(ii) Show that this algorithm generates a Markov Chain $\left(X^{(t)}\right)_{t=0,1,\ldots}$ whose transition kernel is in detailed balance with π.

(for simplicity, you can assume that $\pi(x) > 0$ for all $x \in \mathbb{R}^p$).

6

Let $K \geqslant 1$ be an integer and $D = \{1, \ldots, K\}^2$. Let $S \in \{-1, 1\}^D$, be random, with distribution

$$\pi(S) \propto \exp\left(-J \sum_{\{i,j\} \in \mathcal{N}} s_i s_j\right),$$

where $J \neq 0$ is a constant, $(S)_i = s_i, i \in D$, and $\mathcal{N}$ is the neighbour relationship on D, i.e., it is the subset of the set of unordered pairs $\{\{i, j\} : i, j \in D, i \neq j\}$ containing $\{i, j\}$ if and only if i and j are vertical or horizontal neighbours. Let

$$X = (X_i)_{i \in D}$$

be random, with conditional distribution $X_i | S \sim N(\mu_i, 1)$, where

$$\mu_i = \left(\sum_{j \in \mathcal{N}_i} s_j\right)^7,$$

$\mathcal{N}_i = \{j \in D : \{j, i\} \in \mathcal{N}\}$ denote the neighbours of $i \in D$. Furthermore, assume that all $X_i, i \in D$, are independent, conditional on S. Suppose we observe a realisation of $X = (X_i)_{i \in D}$.

(a) Write the posterior distribution $\pi(S|X)$ up to a normalizing constant.

(b) Given $i \in D$, which factors of $\pi(S|X)$ depend on S_i?

(c) Derive the full conditionals of S_i, i.e. $\pi_i(s_i | (s_j)_{j \neq i}, X)$, and give the Gibbs sampling algorithm for generating a Markov Chain $(S^{(t)})_{t \geqslant 1}$ with stationary distribution $\pi(S|X)$.

(d) Given a realization $S^{(1)}, \ldots, S^{(T)}$ of the Markov Chain, explain how can you estimate

$$\theta = \mathbb{E}\left[\exp\left(\sum_{i \in D} \sqrt{2 + S_i}\right)\right],$$

where the expectation is taken with respect to the posterior distribution of $S|X$. What theoretical result justifies the use of that particular estimator, and is it applicable here? (you can assume without proof that the Markov Chain is irreducible).

(e) Show that the Markov Chain $(S^{(t)})_{t \geqslant 1}$ is irreducible.

END OF PAPER