UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 9:00 am to 11:00 am

PAPER 34

ACTUARIAL STATISTICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider a portfolio of insurance polices where the claim sizes $X_1, X_2, \ldots$ are independent identically distributed positive random variables, and the number N of claims in one year is independent of the X_i. Derive expressions, in terms of the mean and variance of N and X_1, for the mean and variance of the total amount S claimed on the portfolio in one year. Derive an expression for the moment generating function of S in terms of the probability generating function of N and the moment generating function of X_1.

Now assume that X_1 has an exponential distribution with mean μ.

(a) In portfolio A, suppose that N has a Poisson distribution with mean λ. Find the mean and variance of S.

(b) Consider a second portfolio B, where, given λ, the random variable N has a Poisson distribution with mean λ, and where λ is a random variable with density

$$f(\lambda) = \left(\frac{p}{q}\right)^2 \lambda e^{-p\lambda/q}, \quad \lambda > 0,$$

with $0 < p = 1 - q < 1$. Write down the mean λ_0 of λ. Find the mean and variance of S for portfolio B. Suppose we wish to compare portfolio B with the special case of portfolio A where the value of λ is λ_0. Compare the means and variances of S for these two portfolios.

(c) For portfolio B, find the probability generating function of N. Find the moment generating function of S, and hence show that the distribution of S may be written as a mixture of three distributions. You should specify the three distributions and the mixing proportions.

2 A portfolio has claim sizes $X_1, X_2, \ldots$, which are independent identically distributed positive random variables, and the number of claims in one year is N (independent of the X_i). Let $S = \sum_{i=1}^{N} X_i$. Let S_I^* and $\tilde{S}_I$ be the direct insurer's payout in one year on this portfolio under a quota share reinsurance contract with retained proportion α $(0 < \alpha < 1)$, and under a stop loss reinsurance contract with retention M (> 0), respectively. Write down expressions for S_I^* and $\tilde{S}_I$ in terms of S.

(a) Suppose that S has an exponential distribution with mean μ_S. Write down the expectation and variance of S_I^* in terms of α and μ_S.

Find the expectation and variance of $\tilde{S}_I$ in terms of M and μ_S.

If α is chosen so that $\mathbb{E}\big[S_I^*\big] = \mathbb{E}\big[\tilde{S}_I\big]$, find $\operatorname{var}\big[S_I^*\big] - \operatorname{var}\big[\tilde{S}_I\big]$ in terms of μ_S and M and verify directly that this expression is non-negative.

(b) Now suppose that S is no longer assumed to be exponentially distributed. Consider a reinsurance contract where the direct insurer's payout in one year is $S_I^\dagger = g(S)$, where $0 \leqslant g(x) \leqslant x$ for all $x \geqslant 0$. Suppose also that $\mathbb{E}[S_I^\dagger] = \mathbb{E}[\tilde{S}_I] = c$. By showing that

$$\operatorname{var}\big[S_I^\dagger\big] = \mathbb{E}\big[(g(S) - M)^2\big] - (M - c)^2,$$

show that the variance of $S_I^\dagger$ is at least as large as the variance of the direct insurer's payout under stop loss reinsurance.

3

In a classical risk model, claims arrive in a Poisson process with rate λ (> 0), the premium income rate is c (> 0), and the claim sizes have density function f, moment generating function M and finite mean μ (> 0). Assume that the relative safety loading factor ρ is positive, and that there is a unique positive solution R to $M(r) - 1 = cr/\lambda$.

Show that the probability of ruin $\psi(u)$ with initial capital $u \geqslant 0$ satisfies $\psi(u) \leqslant e^{-Ru}$ for $u \geqslant 0$. Find $\lim_{u\to\infty} \psi(u)$.

You are given that

$$\psi(u) = \frac{\lambda\mu}{c}\int_u^\infty f_I(x)dx + \frac{\lambda\mu}{c}\int_0^u \psi(u-x) f_I(x)dx,$$

where $f_I(x)$ is a probability density function. Show that

$$\lim_{u\to\infty} e^{Ru}\psi(u) = A, \text{ where } A = \frac{\rho}{R\int_0^\infty xe^{Rx} f_I(x)dx}.$$

[Hint: Results from renewal theory may be used without proof provided they are clearly stated. You may assume that $\int_0^\infty e^{Rx} f_I(x)dx = c/\big(\lambda\mu\big)$.]

For a particular choice of claim-size distribution,

$$\psi(u) = ae^{-cu} + be^{-du}, \quad u \geqslant 0,$$

where $a > 0$, $0 < a + b < 1$ and $0 < c < d < \infty$. Find R, A and ρ in terms of a, b, c and d.

4

Describe the Bühlmann model and derive the credibility factor and the credibility premium for this model.

For a particular risk, let X_j be the amount claimed in year j, $j = 1, 2, \ldots$, and suppose that, given θ, the random variables $X_1, X_2, \ldots$ are independent with density

$$f(x \mid \theta) = \left(\frac{1}{\theta}\right)^{\alpha} \frac{x^{\alpha-1} e^{-x/\theta}}{\Gamma(\alpha)}, \quad x > 0,$$

where $\alpha > 0$ is known, and that the prior density of θ is

$$\pi(\theta) = \frac{\lambda^k e^{-\lambda/\theta}}{\theta^{k+1}(k-1)!}, \quad \theta > 0,$$

where k is an integer greater than 2. The claims sizes for years $1, \ldots, n$ have been observed. Find the Bühlmann model credibility estimate for the expected claim size in year $n+1$.

Find the Bayesian estimate under quadratic loss of the expected claim size in year $n+1$ given the claims sizes in years $1, \ldots, n$, and compare this value with the Bühlmann credibility estimate.

[Hint: For $\nu > 0$ and a positive integer m, the function $g(y) = \nu^m e^{-\nu/y} / \big(y^{m+1}(m-1)!\big)$ is a probability density function.]

END OF PAPER