UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 9:00 am to 11:00 am

PAPER 31

PERCOLATION AND RELATED TOPICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Write an essay on self-avoiding walks (SAWs) on infinite graphs. Your essay should include:

1. a statement of any condition you are assuming on these graphs,
2. a definition of a SAW,
3. a definition of the connective constant μ,
4. a proof of the existence of μ (any standard result from analysis should be stated clearly but the proof is not required),
5. an outline of the main steps of the proof that $\mu = \sqrt{2+\sqrt{2}}$ for the hexagonal lattice.

2

What is meant by *oriented percolation* on the square lattice $\mathbb{Z}^2$? Define the critical probability $\vec{p}_c$ of this process, and show that $0 < \vec{p}_c < 1$. [Results for undirected percolation may be used without proofs, but clear statements should be given.]

The vertex $(i,j) \in \mathbb{Z}^2$ is called *even* if $i+j$ is even, and *odd* otherwise. Vertical edges of $\mathbb{Z}^2$ are oriented from the even endpoint to the odd endpoint, and horizontal edges from the odd endpoint to the even endpoint. Each edge is declared *open* with probability p, and *closed* otherwise, different edges receiving independent designations.

Let $\theta(p)$ be the probability that the origin is the endpoint of an infinite open path that is oriented away from the origin. Show that $\theta(p) > 0$ if $1-p$ is sufficiently small and positive.

3

(a) Let μ_1, μ_2 be probability measures on the product space $\{0,1\}^E$, where $|E| = N < \infty$. State Holley's theorem for the stochastic inequality $\mu_1 \leqslant \mu_2$.

State the FKG theorem for positive association, and prove it. [You may use Holley's theorem.]

(b) Let $p \in (0,1)$, $q \in [1,\infty)$. Define the random-cluster measure ϕ_n with parameters p, q on the box $\Lambda_n = [-n,n]^2$ of the square lattice $\mathbb{Z}^2$.

Let A be an increasing event that depends on the states of only finitely many edges. Use the FKG inequality to show that $\phi_{n+1}(A) \leqslant \phi_n(A)$, and deduce the existence of the limit $\phi(A) = \lim_{n\to\infty} \phi_n(A)$. [You may assume that ϕ_n satisfies the conditions of the FKG theorem.]

(c) (continuation) Deduce the existence of the limit $\phi(A)$ for any event A that depends on the states of only finitely many edges (but is not necessarily increasing).

4

(a) Consider bond percolation with parameter p on the square lattice $\mathbb{Z}^2$. Define the percolation probability $\theta(p)$, and show that $\theta(\frac{1}{2}) = 0$. [You may use other results from percolation theory so long as you give careful statements of them.]

(b) Let $G = (V, E)$ be a finite, planar, connected graph, with dual graph $G' = (V', E')$ obtained from G by placing a vertex in each face of G and joining two vertices whenever the corresponding faces share an edge. Let $\phi_{p,q}$ and $\phi'_{p,q}$ be the random-cluster measures on G and G'. Show that $\phi_{p,q}$ and $\phi'_{p',q}$ are dual to one another for some $p' = p'(p,q)$ to be determined.

[You may assume Euler's formula: $|V| - |E| + |F| = k$ for any planar graph $G = (V,E)$, with set F of finite faces, and k connected components.]

(c) Let $0 < p < 1$, $q \geqslant 1$, and let ϕ^1_p and ϕ^0_p be the infinite-volume random-cluster measures on $\mathbb{Z}^2$ with parameters p, q and wired and free boundary conditions, respectively. Show that $\phi^0_\pi(0 \leftrightarrow \infty) = 0$ where $\pi = \sqrt{q}/(1+\sqrt{q})$.

[You may assume that ϕ^0_π, ϕ^1_π form a dual pair of probability measures.]

END OF PAPER