UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 9:00 am to 12:00 pm

PAPER 30

STOCHASTIC CALCULUS AND APPLICATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

For this question, if Z is a continuous local martingale then the notation

$$\mathcal{E}(Z) = e^{Z - \langle Z \rangle/2}$$

denotes the corresponding exponential local martingale. You may use without proof the fact that a positive local martingale is a supermartingale.

(a) Let X be a continuous local martingale with $X_0 = 0$. For any $p, q > 1$, establish the identity

$$\mathcal{E}(X)^p = \mathcal{E}(\sqrt{pq}X)^{1/q}\left(e^{\frac{\sqrt{pq}(\sqrt{pq}-1)}{(q-1)}X}\right)^{\frac{q-1}{q}}.$$

Hence, prove that

$$\mathbb{E}\left[\mathcal{E}(X)_T^p\right] \leqslant \left(\mathbb{E}\left[e^{\frac{\sqrt{pq}(\sqrt{pq}-1)}{(q-1)}X_T}\right]\right)^{\frac{q-1}{q}}$$

for any finite stopping time T.

(b) Let X be a continuous local martingale such that $X_t \to X_\infty$ almost surely. For any $r > 1$, use the identity

$$\mathcal{E}(X) = \mathcal{E}(rX)^{\frac{1}{r^2}}\left(e^{\frac{r}{r+1}X}\right)^{\frac{r^2-1}{r^2}}$$

to prove the inequality

$$\mathbb{E}\left[\mathcal{E}(rX)_\infty\right] \geqslant \left(\mathbb{E}\left[\mathcal{E}(X)_\infty\right]\right)^{r^2}\left(\mathbb{E}\left[e^{rX_\infty/2}\right]\right)^{-2(r-1)}.$$

For the rest of the question, let M be a continuous local martingale such that $M_0 = 0$ and $M_t \to M_\infty$ almost surely. Suppose that

$$\sup_T \mathbb{E}[e^{M_T/2}] < \infty$$

where the supremum is over all finite stopping times.

(c) Use part (a) to show that $\mathcal{E}(aM)$ is a uniformly integrable martingale for all $0 < a < 1$. [Hint: You may want to show that $\inf_{p,q>1} \frac{\sqrt{pq}(\sqrt{pq}-1)}{(q-1)} = \frac{1}{2}$.]

(d) Show that $\mathbb{E}[e^{M_\infty/2}] < \infty$. Hence, use part (b) to show that $\mathcal{E}(M)$ is a uniformly integrable martingale.

2

Consider the stochastic differential equation

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \qquad (*)$$

where W is a scalar Brownian motion and b and σ are given function on $\mathbb{R}$.

(a) What is does it mean to say that equation (*) has a weak solution? a strong solution?

(b) What does it mean to say the equation has the uniqueness in law property? the pathwise uniqueness property?

For the rest of the question, specialise to the case where $\sigma(x) = 1$ for all x and b is continuous and bounded.

(c) Let X be a solution of (*). Show that there exists a strictly increasing function g such that $Y = g(X)$ is a local martingale.

(d) Verify the function g found in part (c) is such that the function $h = g' \circ g^{-1}$ is Lipschitz. Hence, prove that equation (*) has a pathwise unique strong solution. You may use without proof Itô's formula and any standard results on the existence and uniqueness of the solutions of stochastic differential equations as long as they are carefully stated.

3

(a) What does it mean to say that $(X_t)_{t\geqslant 0}$ is a Brownian motion in a filtration $(\mathcal{F}_t)_{t\geqslant 0}$?

(b) Show that if X is a continuous adapted process such that $X_0 = 0$ and

$$\mathbb{E}[e^{\mathrm{i}\theta(X_t - X_s)}|\mathcal{F}_s] = e^{-\theta^2(t-s)/2}$$

for all real θ and $0 \leqslant s \leqslant t$, where $\mathrm{i} = \sqrt{-1}$, then X is a Browian motion in the filtration.

(c) State and prove Lévy's characterisation of Brownian motion in the scalar case. You may use standard results from stochastic calculus, such as Itô's formula, without proof.

(d) Let M be a continuous local martingale such that

$$\langle M \rangle_t = \int_0^t K_s ds$$

where K is a continuous, positive adapted process. Show that there exists a continuous adapted process H and a Brownian motion W such that

$$M_t = M_0 + \int_0^t H_s dW_s$$

for all $t \geqslant 0$.

4

Suppose that X is a continuous semimartingale. Show that there exists a continuous, non-decreasing adapted process R such that

$$\langle X\rangle^{(n)} \to R \text{ uniformly on compacts in probability.}$$

where

$$\langle X\rangle_t^{(n)} = \sum_{k=1}^{\infty}(X_{t\wedge t_k^n} - X_{t\wedge t_{k-1}^n})^2$$

where $t_k^n = k2^{-n}$. You may use the following fact without proof: if X is a uniformly bounded continuous martingale, then there exists an adapted process R such that

$$\langle X\rangle^{(n)} \to R \text{ uniformly in probability.}$$

5

Let

$$X_t = e^{-W_t+t/2}\left(x - \int_0^t e^{W_s-s/2}dB_s\right)$$

where W and B are independent Brownian motions and x is a real constant. Let $\Theta = \tan^{-1}(X)$.

(a) Show that there exists a Brownian motion Z such that

$$d\Theta = \cos(\Theta)dZ.$$

(b) Hence, show that

$$\mathbb{P}(X_t \to \infty) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x).$$

(c) Show that the stochastic integral $\int_0^\infty e^{W_s-s/2}dB_s$ is well-defined and has the Cauchy distribution.

You may use standard results of stochastic calculus, such as Itô's formula, without proof.

6

(a) Suppose there are functions $k_i : \mathbb{R}_+ \to \mathbb{R}$ such that

$$K(s,t) = \sum_{i=1}^{\infty} k_i(s) k_i(t),$$

where the sum converges absolutely for all (s,t). Show that there exists a Gaussian process $(X_t)_{t \geqslant 0}$ with $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(X_s X_t) = K(s,t)$.

(b) In the notation of part (a), find $K(s,t)$ in the case where

$$k_i(t) = \int_0^t h_i(u)\, du$$

and the sequence of functions $(h_i)_i$ is an orthonormal basis of the L^2 space of functions on $\mathbb{R}_+$ which are square-integrable with respect to Lebesgue measure.

(c) Suppose X is a continuous Gaussian process such that $\mathbb{E}(X_t) = 0$ and $\mathbb{E}(X_s X_t) = s$ for all $0 \leqslant s \leqslant t$. Show that X is a Brownian motion.

(d) Let W be a Brownian motion. Show that there is a $c \neq 0$ such that the continuous Gaussian process $\hat{W}$ defined by

$$\hat{W}_t = W_t - \frac{c}{t} \int_0^t W_s ds$$

is a Brownian motion.

END OF PAPER