UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 1:30 pm to 4:30 pm

PAPER 3

REPRESENTATION THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

Throughout k is a fixed field, assumed to be algebraically closed.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let Q be a quiver. What is a *representation* X of Q? Define a *morphism* $\theta : X \to Y$ between two representations X and Y of Q.

(b) Suppose that I is a two-sided nilpotent ideal in a k-algebra A such that A/I is isomorphic to a direct product $k \times k \times \cdots \times k$ of copies of the field k. Show that $I = J(A)$, the radical of A. [Basic facts about the radical of a ring may be assumed, if stated clearly.]

Let $A = kQ$ be the path algebra of a quiver Q. Deduce that if Q has no oriented cycles, then $J(A)$ is the two-sided ideal generated by all arrows in Q. Give an example to show that the condition on Q is needed.

(c) It is given that there are precisely three indecomposable representations (up to isomorphism) of the Kronecker quiver K_1, namely

$$X : (0 \to k), \qquad Y : (k \to 0), \qquad Z : (1_k : k \to k).$$

Determine whether or not there are non-zero morphisms between any pair of these representations. Determine whether there are any non-split short exact sequences with these indecomposable representations at the endpoints.

2

(a) What is the *path algebra* of a quiver Q? Show that in a path algebra, the identity element is given by the sum of all constant paths.

(b) Giving brief justification, find the path algebra of Q, when Q is

(i) the r-loop L_r for $r \geqslant 1$;

(ii) the r-subspace quiver S_r (with r source nodes and one sink node).

In (ii) you should find an explicit isomorphism between kS_r and a certain matrix algebra.

(c) Prove that the 3-subspace quiver S_3 has 12 indecomposable representations up to isomorphism. [You should give the main steps in the argument but do not need to give all the details. You may not appeal to Gabriel's theorem.]

3

(a) Let $A = kQ$ be the path algebra of a quiver Q. Let X be a finite-dimensional (left) A-module. Write down the standard resolution of X. (No proof is required but you should define carefully all the terms appearing in the resolution.) Show that the standard resolution is a projective resolution of X and deduce that A is (left) hereditary.

If $|Q_0| = r$, define the Ringel form $\langle\ ,\ \rangle_Q$ on $\mathbb{R}^r$ and prove that if X has dimension vector $\mathbf{n}$, and Y has dimension vector $\mathbf{m}$, then

$$\dim \operatorname{Hom}_Q(X, Y) - \dim \operatorname{Ext}^1_Q(X, Y) = \langle \mathbf{n}, \mathbf{m} \rangle.$$

(b) Henceforth, modules are assumed to be finite-dimensional. With the usual notational conventions, let $X = (X_i, f_\rho)$ and $Y = (Y_i, g_\rho)$ be representations of an arbitrary quiver Q. Let $\rho : i \to j$ be an arrow of Q. Assume that f_ρ has a non-trivial kernel, and that g_ρ has a non-trivial cokernel. Prove that $\operatorname{Ext}^1_Q(X, Y) \neq 0$.

Deduce the following two statements:

(i) Let X be a representation of a quiver without self-extensions. Then, for any arrow ρ, the map f_ρ has maximal rank. [A vector space map $V \to V'$ has *maximal rank*, provided its rank is as large as possible, namely $\min\{\dim_k V, \dim_k V'\}$, or, equivalently, provided the map is a monomorphism or an epimorphism.]

(ii) Again suppose that X is a representation without self-extensions. Let $\pi = (\rho_1, \ldots, \rho_m)$ be a path from i to j in Q of length $m \geqslant 1$, orientated such that $s(\rho_u) = t(\rho_{u+1})$ $(1 \leqslant u \leqslant m-1)$. Write πX for the image $\rho_1 \cdots \rho_m(X_i)$ in X_j and suppose that $\pi X \neq 0$. Let ρ be an arrow such that $s(\rho) = t(\pi)$ and $(\rho\pi)X = 0$. Then $X_{t(\rho)} = 0$.

4

(a) Define the *dimension*, $\dim X$, of a non-empty, locally closed topological space X. Define what is meant by an *algebraic group*. What does it mean to say that there is an *algebraic action* of an algebraic group on a variety?

Let $\varphi : G \to H$ be a homomorphism of algebraic groups. Show that the kernel, $\ker \varphi$, and image, $\mathrm{im}\varphi$, are closed in G and H, respectively. Show also that

$$\dim \ker \varphi + \dim \mathrm{im}\varphi = \dim G.$$

Let Q be a fixed quiver and let $\mathbf{n}$ be a dimension vector. Define conjugation actions of $\mathrm{GL}(\mathbf{n})$ and of $\mathrm{PGL}(\mathbf{n})$ on the representation space $\mathrm{Rep}_Q(\mathbf{n})$ and briefly explain why these actions are algebraic.

(b) What does it mean for an algebraic group to be (i) *connected*, (ii) *unipotent* or (iii) *reductive*? Let X be a finite-dimensional representation of a quiver Q. Show that the automorphism group A of X is a connected linear algebraic group. Show also that A is a semi-direct product of U, a closed normal unipotent subgroup, and a certain direct product of general linear groups.

5

Let $\mathbf{n}$ be a given tuple of non-negative integers and let Q be a quiver. Let $A = kQ$ be the path algebra. Denote the representation space of Q by $\mathrm{Rep}_Q(\mathbf{n})$. Denote the orbit of a representation X of Q under the action of $\mathrm{GL}(\mathbf{n})$ by $\mathcal{O}_X$.

(a) Let Q be the Kronecker quiver K_1. If $\mathbf{n} = (n_1, n_2)$, compute $\mathcal{O}_X$ for any representation X with dimension vector $\mathbf{n}$.

(b) Show that, with the usual notation, the orbit $\mathcal{O}_X$ is precisely the isoclass of the representation X of dimension vector $\mathbf{n}$, namely

$$\mathcal{O}_X = \{x \in \mathrm{Rep}_Q(\mathbf{n}) : X_x \cong X\}.$$

(c) Show that if $0 \to X \to Y \to Z \to 0$ is a non-split short exact sequences of representations, then

$$\dim \mathcal{O}_{X\oplus Z} < \dim \mathcal{O}_Y.$$

(d) Show that, if Q has no oriented cycle, then every orbit closure contains the origin.

6

Let Q be a quiver with vertex set Q_0.

(a) What does it mean to say that a finite-dimensional representation X of Q is a *brick*? Prove that if X is a brick then X is indecomposable.

Give an example of a non-simple local algebra A and an indecomposable A-module which is not a brick.

Assume that the Tits form q_Q is positive definite. Assuming Ringel's lemma, prove the following statements:

(i) Every indecomposable representation of Q is a brick.

(ii) The dimension vectors of the indecomposable representations are precisely those $\mathbf{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\mathbf{n}) = 1$.

(iii) Every indecomposable representation is uniquely determined by its dimension vector, up to isomorphism.

(iv) Q is of finite representation type.

(b) Recall that a *positive root* of Q is a tuple $\mathbf{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\mathbf{n}) = 1$. Let Q be a quiver whose underlying graph is of type A_m (so that there are $m \geqslant 1$ vertices and $m - 1$ undirected edges). How many positive roots does Q have? Write down all positive roots if Q has type A_3.

END OF PAPER