UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2015 1:30 pm to 4:30 pm

PAPER 29

ADVANCED PROBABILITY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove Doob's upcrossings lemma for a discrete-time martingale. State, without giving a proof, the almost sure martingale convergence theorem.

(ii) What does it mean to say that a martingale is uniformly integrable?

(iii) Suppose $X = (X_n \colon n \geqslant 0)$ is a uniformly integrable martingale. Show that, for any stopping time T, the process $X^T = (X_{n \wedge T} \colon n \geqslant 0)$ is uniformly integrable.

2

(i) What does it mean to say that a process $X = (X_t \colon t \geqslant 0)$ is a cadlag martingale? What does it mean to say that a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ satisfies the usual hypotheses.

(ii) Let T be a bounded stopping time, and let X be a cadlag martingale. Show that the stopped process X^T is adapted.

(iii) Let S and T be bounded stopping times with $S \leqslant T$. Let $A \in \mathcal{F}_S$ be given, and set

$$U = S\mathbf{1}(A) + T\mathbf{1}(A^c).$$

Show that U is a stopping time.

(iv) Let $X = (X_t \colon t \geqslant 0)$ be a cadlag integrable process, adapted to a filtration $(\mathcal{F}_t)_{t \geqslant 0}$ satisfying the usual hypotheses. Suppose X_T is integrable with

$$\mathbb{E}[X_T] = 0$$

for every bounded stopping time T. Prove that X is a martingale.

3

Let $B = (B_t \colon t \geqslant 0)$ be a standard Brownian motion on $\mathbb{R}$.

A point $x \in \mathbb{R}$ is said to be a *point of local maximum* of a function $f \colon \mathbb{R} \to \mathbb{R}$ if there exists a $\delta > 0$ such that $f(s) \leqslant f(t)$ for all $s \in [t-\delta, t+\delta] \setminus \{t\}$, and a *point of strict local maximum* if $f(s) < f(t)$ for all $s \in [t-\delta, t+\delta] \setminus \{t\}$.

(i) Show that, almost surely, the set of times of local maxima of B is dense in $[0, \infty)$.

(*Hint: Show first that there is no interval* $[a, b]$ *on which Brownian motion is monotone.*)

(ii) Show that, almost surely, all local maxima for B are strict.

(*Hint: Argue that it is enough to show that events of the form*

$$E(t_1, t_2, t_3, t_4) = \left\{ \max_{t_3 \leqslant t \leqslant t_4} B_t - \max_{t_1 \leqslant t \leqslant t_2} B_t \neq 0 \right\}$$

where $0 \leqslant t_1 < t_2 < t_3 < t_4 < \infty$ *are rational, have probability* 1.)

4

(i) Formulate Kakutani's probabilistic solution to the Dirichlet problem

$$\begin{cases} \Delta u(x) = 0, & x \in D \\ u(\xi) = f(\xi), & \xi \in \partial D \end{cases},$$

carefully stating what is required of the given function f and the domain $D \subset \mathbb{R}^d$.

(ii) Give a proof that the probabilistic construction you have described in (i) provides a solution to the Dirichlet problem. (You may use standard facts about harmonic functions without proof.)

5

(i) Let $X = (X_n : n \geqslant 0)$ be a martingale in discrete time, satisfying

$$|X_{n+1} - X_n| \leqslant M < \infty.$$

Consider the events

$$\mathcal{C} = \left\{ \lim_n X_n \text{ exists and is finite} \right\}$$

and

$$\mathcal{D} = \left\{ \limsup_n X_n = \infty \text{ and } \liminf_n X_n = -\infty \right\},$$

and show that $\mathbb{P}(\mathcal{C} \cup \mathcal{D}) = 1$.

(*Hint: You may wish to consider stopping times of the form*

$$T_A = \inf\{n \geqslant 0 \colon X_n < -A\}.)$$

(ii) Show, by exhibiting a counterexample, that the conclusion above becomes false if it is only assumed that

$$|X_{n+1} - X_n| < \infty.$$

(*Hint: You may find it helpful to consider a sequence* (Y_n) *of independent random variables having*

$$\mathbb{P}(Y_n = 1) = \frac{1}{2^n} \quad \text{and} \quad \mathbb{P}(Y_n = 0) = 1 - \frac{1}{2^n}.)$$

6

(i) State the definition of a Lévy process. What form does the characteristic function of a Lévy process take?

(ii) Check that the Poisson process, as defined in the lectures, satisfies the requirements of a Lévy process.

(iii) Let $X = (X_t \colon t \geqslant 0)$ be a stochastic process. Suppose $(X^n)_{n=1}^{\infty}$ is a sequence of Lévy processes such that, for each t,

$$X_t^n \to X_t \quad \text{in probability}$$

and

$$\lim_n \limsup_{t \downarrow 0} \mathbb{P}(|X_t^n - X_t| > \epsilon) = 0$$

for each $\epsilon > 0$.

Prove that X is a Lévy process.

END OF PAPER