UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 9:00 am to 12:00 pm

PAPER 28

ALGEBRAIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let S be a set of rational primes. Define what it means for the set S to have Dirichlet density δ.

Let S be the set of primes p satisfying $p \equiv 1 \bmod 4$ and $2 \in (\mathbb{F}_p^\times)^4$. Calculate the Dirichlet density of S.

2

Let K be a number field, and let $\mathfrak{b}$ be a fractional ideal of $\mathcal{O}_K$, S_∞ the set of infinite places of K. Define the theta function $\Theta(y, \mathfrak{b})$ for $y \in (\mathbb{R}_{>0})^{S_\infty}$, and state and prove its functional equation.

[You may use the Poisson summation formula and any identity of Fourier transforms provided that you state them precisely.]

Show that

$$\lim_{y \to 0} \|y\|^{1/2} \Theta(y, \mathfrak{b}) = 1,$$

where $\|y\| = \prod_{v \in S_\infty} y_v^{[K_v:\mathbb{R}]}$.

3

Let E/K be an extension of number fields. Define the different ideal $\mathfrak{d}_{E/K}$, and state a theorem relating its prime factorization to ramification in the extension E/K.

Let $m \geqslant 2$ be a square-free integer not divisible by 3 and satisfying $m \not\equiv \pm 1 \bmod 9$. Let $K = \mathbb{Q}(\alpha)$, where $\alpha^3 = m$. Show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$.

4

Let K be a number field. What is a divisor of K? Define the multiplicity $m_v(\mathfrak{c})$ and the generalized ideal class group $H_\mathfrak{c}$ associated to a place v and a divisor $\mathfrak{c}$ of K. Write down two short exact sequences relating $H_\mathfrak{c}$ to the usual ideal class group $\mathrm{Cl}(\mathcal{O}_K)$ and the group $\mathcal{O}_K^\times$ of units of K.

Now let $K = \mathbb{Q}(\sqrt{15})$, let ∞_1, ∞_2 be the infinite places of K, and let $\mathfrak{c}$ be the divisor $\mathcal{O}_K \cdot \{\infty_1, \infty_2\}$ of K. Show that $\# H_\mathfrak{c} = 4$.

Decide whether $H_\mathfrak{c} \cong \mathbb{Z}/4\mathbb{Z}$ or $H_\mathfrak{c} \cong (\mathbb{Z}/2\mathbb{Z})^2$.

[You may use the facts that Cl($\mathcal{O}_K$) $\cong \mathbb{Z}/2\mathbb{Z}$ is generated by the ideal $(2, 1+\sqrt{15})$ and $\mathcal{O}_K^\times$ is generated by -1 and $\epsilon = 4 + \sqrt{15}$.]

END OF PAPER