UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 9:00 am to 12:00 pm

PAPER 27

ELEMENTARY METHODS IN ANALYTIC NUMBER THEORY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Let $\mathcal{P}$ be any set of primes, and let $2 \leqslant \sqrt{D}$.

Let (ρ_d) be any sequence of real numbers subject to the following constraints:

$$\rho_1 = 1, \quad \text{and } \rho_d = 0 \text{ unless } (p \mid d \Rightarrow p \in \mathcal{P}), \quad \text{and } \rho_d = 0 \ \forall d > \sqrt{D}.$$

Show that if we define

$$\lambda_d := \sum_{[d_1,d_2]=d} \rho_{d_1}\rho_{d_2} \quad \forall d,$$

where $[d_1, d_2]$ denotes least common multiple, then the λ_d are upper bound sieve weights (in other words $\sum_{d|n} \lambda_d \geqslant \mathbf{1}_{p|n \Rightarrow p \notin \mathcal{P}}$) of level D.

(b) Let $\mathcal{A} = (a_n)$ be a finite sequence of non-negative numbers, let $X > 0$, and suppose $g(d)$ is a multiplicative function that is supported on squarefree d, and satisfies $0 < g(p) < 1$ for all primes p.

Define the remainder numbers $r(d)$ corresponding to $\mathcal{A}, g, X$. Then show that for any weights λ_d constructed as in part (a), we have

$$\sum_{n:p|n \Rightarrow p \notin \mathcal{P}} a_n \leqslant X\Sigma + \sum_{d \leqslant D} \lambda_d r(d),$$

where $\Sigma := \sum_{t \leqslant \sqrt{D}} \left(\sum_{\substack{m \leqslant \sqrt{D}, \\ t|m}} g(m)\rho_m \right)^2 \prod_{p|t} \left(\frac{1}{g(p)} - 1 \right)$.

(c) For each odd prime p, let $n(p)$ denote the smallest natural number that is a quadratic non-residue mod p.

State the Variance Version of the Large Sieve Inequality, and use it to prove that for any small $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that

$$\#\{3 \leqslant p \leqslant N : p \text{ prime}, \ n(p) > N^{\epsilon}\} \leqslant C(\epsilon) \quad \forall N.$$

[You may assume that for any $\epsilon > 0$, the quantity of N^{ϵ}-smooth numbers less than N is $\geqslant \kappa(\epsilon)N$, where $\kappa(\epsilon) > 0$ is a small constant.]

2

(a) Explain what it means for points $\theta_1, ..., \theta_R \in \mathbb{R}$ to be *δ-spaced*, where $\delta > 0$.

Prove the Exponential Sums Version of the Large Sieve inequality, which states that if $\theta_1, ..., \theta_R$ are δ-spaced, and if $M \in \mathbb{Z}$ and $(a_n)_{M<n\leqslant M+N}$ are any complex numbers, then

$$\sum_{r=1}^{R} \left| \sum_{M<n\leqslant M+N} a_n e(n\theta_r) \right|^2 \leqslant \left(\frac{1}{\delta} + 2\pi N \right) \sum_{M<n\leqslant M+N} |a_n|^2,$$

where $e(\theta) := e^{2\pi i\theta}$ is the complex exponential.

[You may assume the Sobolev–Gallagher inequality provided you state it clearly, but you should prove any other statements that you use.]

(b) Explain *briefly* how you would modify the proof in part (a) to show that for any $\Delta > 0$, and any points $\theta_1, ..., \theta_R \in \mathbb{R}$ (without any spacing condition), we have

$$\sum_{r=1}^{R} \left| \sum_{M<n\leqslant M+N} a_n e(n\theta_r) \right|^2 \leqslant K(\Delta) \left(\frac{1}{\Delta} + 2\pi N \right) \sum_{M<n\leqslant M+N} |a_n|^2,$$

where

$$K(\Delta) := \max_{x\in\mathbb{R}} \#\{1 \leqslant r \leqslant R : ||\theta_r - x|| \leqslant \frac{\Delta}{2}\},$$

and $|| \cdot ||$ denotes distance to the nearest integer.

(c) Let P be large, and take the set of numbers θ_r to be all the fractions a/p, where $P \leqslant p \leqslant 2P$ is prime and $1 \leqslant a \leqslant p-1$. By choosing $\Delta = 1/P$ in part (b), show that if $P \geqslant N$ then

$$\sum_{\substack{P\leqslant p\leqslant 2P,\\ p \text{ prime}}} \sum_{(a,p)=1} \left| \sum_{M<n\leqslant M+N} a_n e(an/p) \right|^2 \ll \frac{P^2}{\log P} \sum_{M<n\leqslant M+N} |a_n|^2.$$

[You may assume any standard estimates for primes provided you state them clearly.]

How does this compare with the bound you could obtain directly from part (a)?

3

(a) Let $\mathcal{P}$ be any set of primes, and let $2 \leqslant \sqrt{D}$. Let $g(d)$ be a multiplicative function supported on squarefree d, and satisfying $0 < g(p) < 1$ for all primes p.

Let (ρ_d) be any sequence of real numbers subject to the following constraints:

$$\rho_1 = 1, \quad \text{and } \rho_d = 0 \text{ unless } (p \mid d \Rightarrow p \in \mathcal{P}), \quad \text{and } \rho_d = 0 \ \forall d > \sqrt{D}.$$

Finally, let

$$\Sigma := \sum_{t \leqslant \sqrt{D}} \Bigg(\sum_{\substack{m \leqslant \sqrt{D}, \\ t \mid m}} g(m)\rho_m \Bigg)^2 \prod_{p \mid t} \left(\frac{1}{g(p)} - 1 \right).$$

Show that the minimum possible value of Σ over all such sequences (ρ_d) is equal to $1/J$, where $J := \sum_{\substack{d \leqslant \sqrt{D}, \\ d \text{ squarefree}, \\ p \mid d \Rightarrow p \in \mathcal{P}}} \prod_{p \mid d} \frac{g(p)}{1-g(p)}$, and that this is attained when

$$\rho_d = (\mathbf{1}_{p \mid d \Rightarrow p \in \mathcal{P}}) \mu(d) \left(\prod_{p \mid d} \frac{1}{1 - g(p)} \right) \frac{1}{J} \sum_{\substack{t \leqslant \sqrt{D}/d, \\ t \text{ squarefree}, (t,d)=1, \\ p \mid t \Rightarrow p \in \mathcal{P}}} \prod_{p \mid t} \frac{g(p)}{1 - g(p)} \quad \forall d \leqslant \sqrt{D}.$$

Show also that this choice of (ρ_d) satisfies $|\rho_d| \leqslant 1$ for all squarefree d.

[You may assume a version of Möbius inversion that will allow you to diagonalise Σ, provided you state it clearly.]

(b) Let $\omega(d)$ denote the number of distinct prime factors of d, and let $\Psi(t) := \sum_{n \leqslant t} \Lambda(n)$. Show that for any large D we have

$$\sum_{\sqrt{D} \leqslant d \leqslant D} 3^{\omega(d)} \ll \frac{1}{\log D} \sum_{m \leqslant D} 3^{\omega(m)} \Psi(D/m).$$

Deduce that $\sum_{\sqrt{D} \leqslant d \leqslant D} 3^{\omega(d)} \ll D \log^2 D$ for all large D.

[You may assume any standard estimates for sums over primes, provided you state them clearly.]

4

(a) State and prove Vaughan's Identity for $\Lambda(n)$.

[You should briefly prove any other identities that you use in your proof.]

State the Bombieri–Vinogradov theorem, and describe *briefly* how it is proved, remarking on the roles of Vaughan's Identity and of the other main ingredients in the proof.

(b) Let $\pi(x;q,a)$ denote the number of primes less than x that are $\equiv a \bmod q$, and let $\omega(q)$ denote the number of distinct prime factors of q. Show that for any large x and any $Q \leqslant x^{0.99}$ we have

$$\sum_{q \leqslant Q} 3^{\omega(q)} \max_{(a,q)=1} \left| \pi(x;q,a) - \frac{1}{\phi(q)} \int_2^x \frac{dt}{\log t} \right|$$

$$\ll \sqrt{\sum_{q \leqslant Q} \max_{(a,q)=1} \left| \pi(x;q,a) - \frac{1}{\phi(q)} \int_2^x \frac{dt}{\log t} \right|} \sqrt{\frac{x}{\log x} \sum_{q \leqslant Q} \frac{9^{\omega(q)}}{\phi(q)}}.$$

[You may assume a standard sieve upper bound for $\pi(x;q,a)$ provided you state it clearly.]

(c) By applying Selberg's upper bound sieve to the sequence of values $p+2$, where $p \leqslant x$ runs over primes, and using parts (a) and (b) to control the remainder sum, prove that

$$\#\{p \leqslant x : p, p+2 \text{ are both prime}\} \ll \frac{x}{\log^2 x}.$$

[You may assume that $\sum_{q \leqslant Q} \frac{9^{\omega(q)}}{\phi(q)} \ll \log^9 Q$ for all large Q, and you may assume any other standard estimates provided you state them clearly.]

5

(a) State Selberg's upper bound sieve, and use it to prove that for any $1000 \leqslant z \leqslant x$ we have

$$\#\{x < n \leqslant x + z : p \mid n \Rightarrow p \equiv 1 \bmod 4\} \ll \frac{z}{\sqrt{\log z}}.$$

[You may assume that $\sum_{p \leqslant x, p \equiv 1 \bmod 4} \frac{1}{p} = (1/2) \log\log x + c + O(1/\log x)$, where c is a constant, and you may assume any other standard estimates for sums over primes provided you state them clearly.]

(b) Let $T(x) := \sum_{n \leqslant x} \Psi(x/n)$, where $\Psi(t) := \sum_{m \leqslant t} \Lambda(m)$. Show that $T(x) = \sum_{n \leqslant x} \log n$, and deduce that

$$T(x) = x \log x - x + O(\log(x+1)) \quad \forall x \geqslant 1.$$

By considering $\sum_{n \leqslant x} \Psi(x/n) \sum_{d|n} \lambda_d$ for a suitable choice of weights λ_d, prove that

$$\Psi(x) + \sum_{n \leqslant x} \Psi(x/n) \frac{\Lambda(n)}{\log x} = 2x + O\left(\frac{x}{\log x}\right) \quad \forall x \geqslant 2.$$

[You may assume any standard identities involving the Möbius function, and the estimates of Chebychev and Mertens for sums over primes, provided you state them clearly. You may also assume that $\sum_{n \leqslant x} 1/n = \log x + \gamma + O(1/x)$ for all $x \geqslant 1$, where γ is Euler's constant.]

END OF PAPER