UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 9:00 am to 11:00 am

PAPER 26

LOCAL FIELDS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove a version of Hensel's Lemma, and use it to compute the groups $\mathbb{Z}_p^*/(\mathbb{Z}_p^*)^2$ for p a prime.

(ii) State the Hasse-Minkowski Theorem. Suppose that p, $p+12$ and $p+24$ are primes. Let

$$Q(x, y, z) = px^2 - (p+12)y^2 + (p+24)z^2.$$

Show that $Q(x, y, z) = 0$ has a solution in integers x, y, z (not all zero) if and only if $p \equiv \pm 1 \pmod 8$.

[*It may help to note that* $Q(x, 1, 2) = px^2 - (p + 8m)$ *for some integer* m.]

2

(i) Let K be a finite extension of $\mathbb{Q}_p$. Show that if $|\cdot|_p$ on $\mathbb{Q}_p$ extends to an absolute value $|\cdot|$ on K then this extension is unique, and $(K, |\cdot|)$ is complete. [*You may assume* $\mathbb{Z}_p$ *is compact.*]

(ii) Let K be a number field with ring of integers $\mathcal{O}_K$. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$ with $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$. Prove that the completion of K with respect to the $\mathfrak{p}$-adic absolute value is a finite extension of $\mathbb{Q}_p$ and that every finite extension of $\mathbb{Q}_p$ arises in this way.

3

Let $(K, |\cdot|)$ be a non-archimedean valued field. Define the valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}$, and residue field k. Suppose that K is locally compact. Show that $\mathfrak{m}$ is principal, say $\mathfrak{m} = (\pi)$, and k is finite, say $|k| = q$. Prove the following.

(i) If $a \in \mathcal{O}_K^*$ then (a^{q^n}) converges in K and its limit is a $(q-1)^{\text{th}}$ root of unity which is congruent to a mod π.

(ii) If $\operatorname{char} K = 0$ and r is sufficiently large then $(1 + \pi^r \mathcal{O}_K, \times) \cong (\mathcal{O}_K, +)$.

Hence determine the number of roots of unity in $\mathbb{Q}_p$. In the case $K = \mathbb{Q}_2(i)$, where $i = \sqrt{-1}$, show that $\mathcal{O}_K^* \cong \mu_4 \times \mathcal{O}_K$. How may quadratic extensions of K are there (up to isomorphism)?

4

Write an essay on higher ramification groups, and illustrate by computing these groups for $\mathbb{Q}_2(\zeta_8)/\mathbb{Q}_2$ and $\mathbb{Q}_3(\zeta_3, \sqrt[3]{2})/\mathbb{Q}_3$. [*You should carefully state any results you need about Eisenstein polynomials.*]

END OF PAPER