UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 1:30 pm to 4:30 pm

PAPER 25

COMPUTABILITY AND LOGIC

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet

Treasury Tag

Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What are *many-one reducibility* and *Turing-reducibility*?

State and prove the Friedberg–Muchnik theorem.

2

(i)

What is an immune set? What are incompressible strings? Show that the set of incompressible strings is immune.

(ii)

Explain how Friedman's finite form (FFF) of Kruskal's theorem follows from Kruskal's theorem. What is the significance of FFF?

3

Prove carefully that if f is a function $\mathbb{N}^k \to \mathbb{N}$ declared by primitive recursion then there is a formula $\phi(y, \vec{x}, \vec{z})$ in the language with 0, 1, $+$, $\times$, $<$ and $=$ ("the language of ordered rings"), containing no unrestricted quantifiers, such that $y = f(x_1 \ldots x_k)$ iff $(\exists \vec{z})\phi(y, \vec{x}, \vec{z})$

4

Show that every partial computable function can be represented by a λ-term acting on Church numerals.

5

What is the Goodstein function? Prove that it is total. Explain why this gives rise to a decidable wellordering of $\mathbb{N}$ of length ϵ_0.

6

What is a recursively axiomatisable theory? Establish that any sufficiently strong sound recursively axiomatised arithmetic of the natural numbers is incomplete. What is a productive set? Establish that the set of arithmetic truths is productive.

END OF PAPER