UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 9:00 am to 12:00 pm

PAPER 24

TOPICS IN SET THEORY

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define carefully any two of the following three italicized concepts or statements:

(i) the partial order $\mathbb{P} = (P, <_{\mathbb{P}})$ is *well-founded*;

(ii) the sets A and B are a *Sierpiński decomposition* of the plane;

(iii) the subset $C \subseteq {}^{\omega}\omega$ is *dominating* in $({}^{\omega}\omega, \preceq^*)$.

(b)

(i) Show that if D is a dense subset of the linear order $(\mathbb{R}, <)$ of real numbers and $\varphi : D \to D$ is an order-automorphism, then φ has a unique extension to an order-automorphism $\tilde{\varphi}$ of $(\mathbb{R}, <)$.

(ii) Prove that any order-automorphism of $(\mathbb{R}, <)$ is determined by its values on $\mathbb{Q}$. Deduce that there are exactly $2^{\aleph_0}$ order-automorphisms of $(\mathbb{R}, <)$.

(iii) Prove that there exists a rigid dense subset X of the real line such that $|X| = 2^{\aleph_0}$ and X has no non-trivial order-automorphism, i.e. no order-automorphism other than the identity.

(c) Let κ be an infinite cardinal. Using a well-ordering of $[\kappa]^{\omega}$ or otherwise, prove that

$$\kappa \not\to (\omega)^{\omega}_{2}$$

(Recall that the notation $\lambda \to (\mu)^{\alpha}_{\beta}$, where $\lambda, \mu, \alpha, \beta$ are ordinals, means: for every function $f : [\lambda]^{\alpha} \to \beta$, there exists $X \in [\lambda]^{\mu}$ such that $f \restriction [X]^{\alpha}$ is constant.)

2

(a) Define carefully any <u>two</u> of the following three italicized concepts:

(i) the partial order $\mathbb{X} = (X, <_{\mathbb{X}})$ is a *Suslin line*;

(ii) the tree $\mathbb{T} = (T, <_{\mathbb{T}})$ is a *Kurepa tree*;

(iii) the function $f : \delta \to \alpha$ is a *cofinal map* (in α).

(b) Let κ be an infinite cardinal. Suppose for each ordinal $\delta < \kappa^+, C_\delta \subseteq \delta$ is closed and unbounded in δ and has order-type at most κ. For each ordinal $\beta < \kappa^+$, define the sequences $\langle \beta_0^\alpha, \ldots, \beta_n^\alpha \rangle$ and the functions $\rho_\beta : \beta \to [P(\beta)]^{<\omega}$ as follows.
For $\alpha < \beta$, let $\beta_0^\alpha = \beta, \beta_{i+1}^\alpha = min(C_{\beta_i^\alpha} \setminus \alpha) < \beta_i^\alpha$ until $\beta_n^\alpha = \alpha$.
Set $\rho_\beta(\alpha) = \langle C_{\beta_i^\alpha} \cap \alpha : i < n \rangle$.
Of the following three properties, prove (iii) and any other one of (i) and (ii).

(i) If $\xi < \alpha < \beta < \kappa^+$, then there exists a unique $j \in \omega$ such that for $(\forall i \leqslant j)(\beta_i^\xi = \beta_i^\alpha)$ and $\xi \leqslant \beta_{j+1}^\xi < \alpha$.

(ii) If $\xi < \alpha < \beta < \kappa^+$ and $\rho_\beta(\xi)$ and $\rho_\beta(\alpha)$ are of the same length n, then $(\exists j < n)(C_{\beta_j^\xi} \cap \xi$ is a proper initial segment of $C_{\beta_j^\alpha} \cap \alpha)$.

(iii) If $\alpha < \beta < \gamma$ and $\rho_\beta(\alpha) = \rho_\gamma(\alpha)$ then $\rho_\beta \restriction \alpha = \rho_\gamma \restriction \alpha$.

(c) Prove that the Continuum Hypothesis CH implies the existence of an $\aleph_2$–Aronszajn tree.

3

(a) Define carefully any <u>two</u> of the following three italicized concepts or statements:

(i) the *diagonal intersection* of the family $\{C_\alpha \subseteq \delta : \alpha < \delta\}$ for a limit ordinal δ;

(ii) *Fodor's Lemma* for a regressive function $f : S \subset \lambda \to \lambda$;

(iii) the *combinatorial principle* $\clubsuit$.

(b) Suppose S is a stationary subset of ω_1.

(i) Prove that $\diamondsuit_S$ implies the existence of a non-special Aronszajn tree. Your proof should cover the main points of the argument.

(ii) Deduce that the failure of Suslin's Hypothesis is consistent relative to ZFC.

(c) Let KH be the assertion: there exists a family $\mathcal{F} \subseteq P(\omega_1)$ such that $|\mathcal{F}| = \aleph_2$ and $(\forall \alpha < \omega_1)(\{X \cap \alpha : X \in \mathcal{F}\}$ is countable). For $X \in \mathcal{F}$, let $f_X : \omega_1 \to P(\omega_1)$ be defined as follows: $f_X(\alpha) = X \cap \alpha$.

(i) Suppose that there exists a Kurepa tree $\mathbb{T} = (T, \leqslant_{\mathbb{T}})$ tree such that $T = \omega_1$ and $\alpha <_{\mathbb{T}} \beta \Rightarrow \alpha < \beta$. Show that the assertion KH holds.

(ii) Using the functions $\{f_X : X \in \mathcal{F}\}$ or otherwise, prove that KH implies the existence of a normal $(\aleph_1, \aleph_1)$–tree possessing at least $\aleph_2$ branches of length $\aleph_1$.

4

(a) Define carefully any <u>two</u> of the following three italicized concepts or statements:

(i) the structure $\mathbb{M} = (M, E^{\mathbb{M}})$ is a *standard model of ZFC*;

(ii) the *Mostowski Collapse theorem*;

(iii) the formula $\varphi(v_1, \ldots, v_n)$ with free variables amongst $v_1, \ldots, v_n$ in the language of ZFC is *absolute*.

(b)

(i) Suppose that R is a binary relation on the class X. Prove that R is well-founded if, and only if, there exists a function $\rho : X \to Ord$ such that if xRy (i.e. $\langle x, y\rangle \in R$), then $\rho(x) < \rho(y)$.

(ii) Show that the predicate"R is a well-founded binary relation" is absolute.

(iii) Let $\mathbf{R}$ be a new predicate symbol distinct from $\in$; for $\alpha \in Ord$ and $X \subseteq V_\alpha$, let $\langle V_\alpha, \in, X\rangle$ be the expansion of the standard structure $(V_\alpha, \in)$ in which $\mathbf{R}$ is interpreted as X. Suppose that κ is an inaccessible cardinal and $R \subseteq V_\kappa$. Show that the set $\{\alpha < \kappa : \langle V_\alpha, \in, R \cap V_\alpha\rangle$ is an elementary submodel of $\langle V_\kappa, \in, R\rangle\}$ is closed unbounded in κ.

(c) A cardinal κ is *Mahlo* if κ is inaccessible and $\{\alpha < \kappa : \alpha \text{ is inaccessible}\}$ is stationary in κ.

(i) Suppose that κ is an inaccessible cardinal. Prove that κ is Mahlo if, and only if, for any $R \subseteq V_\kappa$, there is an inaccessible cardinal $\alpha < \kappa$ such that the structure $\langle V_\alpha, \in, R \cap V_\alpha\rangle$ is an elementary submodel of $\langle V_\kappa, \in, R\rangle$.

(ii) Showing first that the predicate "v is inaccessible" is Π_1 or otherwise, prove that if κ is Mahlo (in V), then $\mathbb{L} \models (\kappa \text{ is Mahlo})$, where $\mathbb{L}$ is the universe of constructible sets.

5

(a) Suppose that $\mathbb{M}$ is a countable transitive model of ZFC. Choose two of the following three outcomes, and for each of your choices, describe a forcing that achieves the desired outcome in a $\mathbb{P}$–generic extension $\mathbb{M}[G]$.

(i) the inaccessible cardinal $\kappa \in M$ is collapsed to $\aleph_1$ while all cardinals above κ are preserved;

(ii) the continuum has size $\aleph_3$ (in $\mathbb{M}[G]$) while if $f \in M[G]$ is a function from an ordinal α into Ord, then there is a function $F \in M$ such that $dom(F) = \alpha, (\forall \xi < \alpha)(|F(\xi)| = \aleph_0)$, and $range(f) \subseteq \bigcup_{\xi<\alpha} F(\xi)$;

(iii) the combinatorial principle $\diamondsuit$ holds in $\mathbb{M}[G]$.

(b) Suppose G is $\mathbb{P}$–generic over $\mathbb{M}$, where $\mathbb{P}$ is a partial order in a countable transitive model $\mathbb{M}$ of ZFC. Prove that the Axiom Schema of Separation for a formula $\varphi(u, w, v_1, \ldots, v_n)$ holds in the generic extension $\mathbb{M}[G]$.

(c)

(i) Let $\mathbb{P}$ be a countable separative partial order and let G be $\mathbb{P}$–generic over a countable transitive model $\mathbb{M}$ of ZFC. Suppose that $Y \subseteq M$ is an uncountable set in $M[G]$.
Show there exists an uncountable $X \in M$ such that

$$\mathbb{M}[G] \models (\check{X} \subseteq \dot{Y}).$$

(ii) Let $\mathbb{M}$ be a countable transitive model of ZFC such that ($\mathbb{T}$ is a normal Suslin tree)$^{\mathbb{M}}$. Suppose $(\mathbb{P} = Fn(\omega, 2, \aleph_0))^{\mathbb{M}}$. Let G be $\mathbb{P}$–generic over $\mathbb{M}$.
Prove:

$$\mathbb{M}[G] \models (\mathbb{T} \text{ is a Suslin tree}).$$

6

(a) Suppose $\mathbb{P}$ is a partial order in the countable transitive model $\mathbb{M}$. Define carefully any <u>two</u> of the following three italicized concepts:

(i) $\mathbb{P}$ *preserves cardinals*;

(ii) the *Definability Lemma* for a formula $\varphi(v_1, \ldots, v_n)$;

(iii) $\mathbb{P}$ is κ–closed for a cardinal $\kappa \in M$.

(b)

(i) Suppose that $\mathcal{D} = \{D_n : n \in \omega\}$ is a family of dense sets in the partial order $\mathbb{P}$ and $p \in P$. Show that there exists a $\mathcal{D}$–generic set $G \subseteq P$ such that $p \in G$.

(ii) Let $\mathbb{M}$ be a countable transitive model of ZFC. Prove there exists a forcing $\mathbb{P} \in M$ such that if $G \subseteq P$ is generic over $\mathbb{M}$, then

$$\aleph_1^{\mathbb{M}} = \aleph_1^{\mathbb{M}[G]};$$

$$({}^{\omega}2)^{\mathbb{M}} = ({}^{\omega}2)^{\mathbb{M}[G]};$$

$$M[G] \text{ contains a surjection from } \aleph_1^{\mathbb{M}[G]} \text{ onto } (\mathcal{P}(\omega))^{\mathbb{M}[G]}.$$

(iii) Using part (ii) or otherwise, prove that the Continuum Hypothesis is consistent relative to ZFC.

(c)

(i) Suppose that $\mathbb{M}$ is a countable transitive model of ZFC and

$\mathbb{M} \models (\kappa = cf(\kappa) > \aleph_0$, and $\mathbb{P}$ is a partial order satisfying the κ-chain condition).

Suppose $p \in P$ and $p \Vdash (\dot{C}$ is a closed unbounded subset of $\kappa)$. Prove that there exists a closed unbounded set $D \in M$ such that

$$p \Vdash (\check{D} \subseteq \dot{C}).$$

(ii) Suppose that λ is an inaccessible cardinal possessing the partition property $\lambda \to (\lambda)^{<\omega}_{\beta}$ for every $\beta < \lambda$, i.e. for any $f : [\lambda]^{<\omega} \to \beta$, then there exists a subset $X \in [\lambda]^{\lambda}$ which is homogeneous for f: for every $n \in \omega, |f[[X]^n]| \leqslant 1$, that is, $f \restriction [X]^n$ is constant (the constant may vary with n).
Prove that if $\mathbb{P}$ is a partial order of size less than λ, then

$$\Vdash_{\mathbb{P}} \lambda \to (\lambda)^{<\omega}_2.$$

END OF PAPER