UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2015 1:30 pm to 3:30 pm

PAPER 23

MODEL THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{L}$ be a language. If F is a filter on a set I (not necessarily an ultrafilter) and for each $i \in I$, $\mathcal{M}_i$ is an $\mathcal{L}$-structure, then we call $\prod_{i\in I} \mathcal{M}_i/F$ the *reduced product by* F.

Recall that Łoś's theorem says that if F is an ultrafilter, then for every formula φ and every finite tuple $f_0, ..., f_k$ of functions from the product, we have

$$\prod_{i\in I} \mathcal{M}_i/F \models \varphi([f_0], ..., [f_k]) \iff \{i \in I\, ;\, \mathcal{M}_i \models \varphi(f_0(i), ..., f_k(i)\} \in F.$$

(a) The class of *tame formulas* is defined as the closure of the atomic formulas under $\wedge$ and $\exists$. Show that Łoś's theorem restricted to tame formulas holds for reduced products (i.e., without assuming that F is an ultrafilter).

(b) Show that Łoś's theorem does not hold in general for reduced products if formulas contain $\vee$.

(c) Show that Łoś's theorem does not hold in general for reduced products if formulas contain $\neg$.

(d) Let κ be an infinite cardinal. Define what it means for an ultrafilter U on a set X to be κ-complete.

(e) Suppose $\mathcal{M} = (M, ...)$ is an $\mathcal{L}$-structure such that the cardinality of M is at least κ. Let U be a κ-complete nonprincipal ultrafilter on κ. Show that the ultrapower of $\mathcal{M}$ by U has more than κ many elements.

2

(a) State precisely the first quantifier elimination test QET1 from the lecture (in words: "Suppose that if two models of T have a common substructure, and any quantifier-free formula with parameters in the substructure has a witness in the first model, then it does in the second. Then T has quantifier elimination").

(b) Derive the following quantifier elimination test QET2 from QET1, giving proper definitions for all terms occurring in the statement:

> Suppose that T has algebraically prime models and for every $\mathcal{M} \subseteq \mathcal{N}$ with $\mathcal{M} \models T$ and $\mathcal{N} \models T$, $\mathcal{M}$ is simply closed in $\mathcal{N}$. Then T has quantifier elimination. (QET2)

(c) Let DAG be the theory of non-trivial torsion-free divisible Abelian groups in the language of groups. This means that it consists of the axioms of Abelian groups, the statement $\exists x(x \neq 0)$, and for each $n \geqslant 1$, the axioms

$$\forall y \exists x(\underbrace{x + \ldots + x}_{n \text{ times}} = y) \text{ and}$$

$$\forall x(x \neq 0 \rightarrow \underbrace{x + \ldots + x}_{n \text{ times}} \neq 0).$$

Show that DAG has algebraically prime models. (In your algebraic constructions of the algebraically prime models, you may assume without proof that your definitions are well-defined.)

(d) Let DAG' be the theory DAG without the non-triviality axiom $\exists x(x \neq 0)$. Prove or refute that the second condition of QET2 holds for DAG', i.e.,

> If $\mathcal{M} \subseteq \mathcal{N}$ such that $\mathcal{M} \models \mathsf{DAG}'$ and $\mathcal{N} \models \mathsf{DAG}'$, then $\mathcal{M}$ is simply closed in $\mathcal{N}$.

Prove or refute that DAG' has quantifier elimination.

3

(a) Define what it means for a theory T to be model complete.

(b) Show that if a theory has quantifier elimination, then it is model complete.

(c) Define what it means for a theory T' to be the model companion of T.

(d) Show that for every theory there is at most one model companion.

(e) Work in the language $\mathcal{L}_{\text{Fields}}$ of fields. Define the theory RCF of real closed fields and the theory FRF of formally real fields and show that RCF is the model companion of FRF.

4

(a) Define the notion of quantifier rank and the notion of n-equivalence of models and use them to precisely state the Ehrenfeucht-Fraïssé theorem including the definition of the game $\mathrm{G}_n(\mathcal{M},\mathcal{N})$.

(b) Consider the following game $\mathrm{G}_{\mathrm{timed}}(\mathcal{M},\mathcal{N})$: player I starts by picking a natural number n, and after this, the two players play the Ehrenfeucht-Fraïssé game $\mathrm{G}_n(\mathcal{M},\mathcal{N})$. Show that player II has a winning strategy in $\mathrm{G}_{\mathrm{timed}}(\mathcal{M},\mathcal{N})$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are elementarily equivalent.

(c) Give an example of structures $\mathcal{M}$ and $\mathcal{N}$ such that player II has a winning strategy in $\mathrm{G}_{\mathrm{timed}}(\mathcal{M},\mathcal{N})$, but not in the infinite Ehrenfeucht-Fraïssé game $\mathrm{G}_\omega(\mathcal{M},\mathcal{N})$.

(d) We consider the language $\mathcal{L}_{\mathrm{rG}} = \{E, \mathsf{root}\}$ of rooted directed graphs and structures $\mathcal{G} = (G, E, r_G)$ and $\mathcal{H} = (H, F, r_H)$. A rooted directed graph is called *connected* if every vertex can be reached by a finite path from the root. A relation $Z \subseteq G \times H$ is called a *bisimulation between $\mathcal{G}$ and $\mathcal{H}$* if $(r_G, r_H) \in Z$ and the following conditions hold:

"back" If $(g,h) \in Z$ and there is some $h' \in H$ such that $(h,h') \in F$, then there is some $g' \in G$ with $(g,g') \in E$ and $(g',h') \in Z$.

"forth" If $(g,h) \in Z$ and there is some $g' \in G$ such that $(g,g') \in E$, then there is some $h' \in H$ with $(h,h') \in E$ and $(g',h') \in Z$.

If there is a bisimulation between $\mathcal{G}$ and $\mathcal{H}$ we call the two rooted graphs *bisimilar*. Give an example of rooted graphs that are bisimilar but not isomorphic.

(e) The *bisimulation game* $\mathrm{G}_{\mathrm{bisim}}(\mathcal{G},\mathcal{H})$ is defined as follows: The game starts in position (r_G, r_H). Each round starts in a position $(v,w) \in G \times H$ and consists of two moves; first, player I chooses either v or w and then picks either some $v' \in G$ such that $(v,v') \in E$ or some $w' \in H$ such that $(w,w') \in F$; after that, player II has to play in the other graph and responds with some $w' \in H$ such that $(w,w') \in H$ or $v' \in F$ such that $(v,v') \in E$. If one of the two players cannot find such an element, the game ends and the player who couldn't move loses. Player II wins if he doesn't lose after any finite number of steps.

Prove that for any two connected rooted directed graphs $\mathcal{G} = (G, E, r_G)$ and $\mathcal{H} = (H, F, r_H)$, player II has a winning strategy in $\mathrm{G}_{\mathrm{bisim}}(\mathcal{G},\mathcal{H})$ if and only if $\mathcal{G}$ and $\mathcal{H}$ are bisimilar.

5

(a) Let $\mathcal{M}$ be a structure and $A \subseteq M$. Suppose that P is an A-type over $\mathcal{M}$ and that $Q \subseteq P$ is finite. Show that Q is realized in $\mathcal{M}$.

(b) Let $\mathcal{M}$ be a structure and $A \subseteq M$. By $\mathrm{S}_n^{\mathcal{M}}(A)$ we denote the set of complete n-A-types over $\mathcal{M}$. Give a definition of the Stone topology on $\mathrm{S}_n^{\mathcal{M}}(A)$.

(c) State the Omitting Types Theorem (giving precise definitions of the terms occurring in it).

(d) Let $\mathcal{M}$ be a structure and $A \subseteq M$. Prove that every isolated A-type over $\mathcal{M}$ is realized in $\mathcal{M}$.

(e) Give counterexamples (i.e., languages $\mathcal{L}$, $\mathcal{L}$-structures $\mathcal{M}$, and sets $A \subseteq M$) to refute the following assertions:

(i) For every A-type P over $\mathcal{M}$ there is a structure $\mathcal{N} \equiv \mathcal{M}$ such that P is realized in $\mathcal{N}$ by infinitely many elements.

(ii) Every A-type can be omitted in some $\mathcal{N} \equiv \mathcal{M}$.

(f) A model $\mathcal{M} \models T$ is called a *prime model for* T if for every model $\mathcal{N} \models T$ there is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$. Show that if $\mathcal{M}$ is a prime model for T, then for every $x \in M$, $\mathrm{tp}_1^{\mathcal{M}}(x/\varnothing)$ is isolated.

END OF PAPER