UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 1 June, 2015 1:30 pm to 4:30 pm

PAPER 22

CATEGORY THEORY

*Attempt no more than **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) State the Yoneda Lemma. Whereas the Yoneda Lemma establishes a certain natural isomorphism, give explicit definitions of the bijective mappings that constitute this isomorphism, as well as the inverses of these mappings.

(b) Let $\mathscr{C}$ be a locally small category, and let $X : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf on X. Give the definition of the *category of elements* of the presheaf X, and define the notion of *universal element* of X. Show that X has a universal element if and only if X is representable.

(c) Let $\mathscr{A}$ and $\mathscr{B}$ be locally small categories categories, and let $H : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathrm{Set}$ be a functor such that for each object B of $\mathscr{B}$ the resulting functor $H(-, B) : \mathscr{A}^{\mathrm{op}} \to \mathrm{Set}$ admits a representation (GB, ψ_B). Show that there is a unique functor $G : \mathscr{B} \to \mathscr{A}$ given on objects by $B \mapsto GB$ such that the given isomorphisms $\psi_B : \mathscr{A}(-, GB) \xrightarrow{\sim} H(-, B)$ are natural in $B \in \mathscr{B}$.

2

Let $\mathscr{A}$ and $\mathscr{C}$ be locally small categories, and suppose that $\mathscr{C}$ is complete. Let $\mathscr{J}$ and $\mathscr{K}$ be small categories.

(a) Let $D : \mathscr{A} \to [\mathscr{J}, \mathscr{C}]$ be a functor, and for each $A \in \mathsf{ob}\,\mathscr{A}$, let $\varprojlim_j (DA)j$ be a limit of $DA : \mathscr{J} \to \mathscr{C}$. Show that there is a functor $L : \mathscr{A} \to \mathscr{C}$ given on objects by $A \mapsto \varprojlim_j (DA)_j$. You may use standard results on representations of functors.

In proving the following you may use any results given in class or in the Examples Sheets.

(b) Letting $F : \mathscr{B} \to \mathscr{C}$ be a functor, show that the following are equivalent.

(i) F preserves limits.

(ii) For each object C of $\mathscr{C}$, the composite functor

$$\mathscr{B} \xrightarrow{F} \mathscr{C} \xrightarrow{\mathscr{C}(C,-)} \mathrm{Set}$$

preserves limits.

(c) Show that for each object C of $\mathscr{C}$, the composite functor

$$[\mathscr{J}, \mathscr{C}] \xrightarrow{\varprojlim} \mathscr{C} \xrightarrow{\mathscr{C}(C,-)} \mathrm{Set}$$

is representable by explicitly exhibiting a representing object.

(d) Show that the functor $\varprojlim : [\mathscr{J}, \mathscr{C}] \to \mathscr{C}$ preserves limits.

(e) By using (d), show that for any functor $D : \mathscr{J} \times \mathscr{K} \to \mathscr{C}$,

$$\varprojlim_{j \in \mathscr{J}} \varprojlim_{k \in \mathscr{K}} D(j,k) \cong \varprojlim_{k \in \mathscr{K}} \varprojlim_{j \in \mathscr{J}} D(j,k)\ .$$

Here, the left- and right-hand-sides are defined as the limits of the composite functors

$$\mathscr{J} \xrightarrow{D^{(1)}} [\mathscr{K}, \mathscr{C}] \xrightarrow{\varprojlim} \mathscr{C}\ ,\quad \mathscr{K} \xrightarrow{D^{(2)}} [\mathscr{J}, \mathscr{C}] \xrightarrow{\varprojlim} \mathscr{C}$$

respectively, where $D^{(1)}$ and $D^{(2)}$ are given by $D^{(1)}j = D(j,-)$ and $D^{(2)}k = D(-,k)$, respectively.

3

(a) Explicitly give the equivalent formulation of the notion of adjunction in terms of a pair of functors equipped with a pair of natural transformations satisfying the *triangular equations* (without proving that this formulation is equivalent to other formulations).

(b) Given a presheaf $X : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ on a locally small category $\mathscr{C}$, show that the following are equivalent.

(i) X is representable.

(ii) X preserves limits, and the forgetful functor $G : \mathsf{Elts}(X) \to \mathscr{C}$ has a colimit.

(c) Let $F : \mathscr{X} \to \mathscr{A}$ be a functor between locally small categories, and suppose that F preserves colimits. State and prove a result to the effect that F has a right adjoint if and only if certain (possibly large) colimits exist in $\mathscr{X}$.

4

(a) State the *General Adjoint Functor Theorem.*

(b) State the *Special Adjoint Functor Theorem.*

(c) Prove the Special Adjoint Functor Theorem. You may invoke the General Adjoint Functor Theorem, and you may use the fact that pullbacks of monomorphisms are monomorphisms.

5

(a) Given an adjunction $K \underset{\varepsilon}{\overset{\rho}{\dashv}} J : \mathscr{C} \to \mathscr{B}$, show that J is fully faithful if and only if ε is an isomorphism.

(b) Let $K \overset{\rho}{\dashv} J : \mathscr{C} \hookrightarrow \mathscr{B}$, where J is the inclusion of a full, replete subcategory $\mathscr{C}$ of $\mathscr{B}$, and let B be an object of $\mathscr{B}$. Show that the following conditions are equivalent:

(i) $B \in \mathsf{ob}\,\mathscr{C}$.

(ii) For every object B' of $\mathscr{B}$, the mapping

$$\mathscr{B}(KB', B) \xrightarrow{\mathscr{B}(\rho_{B'}, B)} \mathscr{B}(B', B)$$

is a bijection.

(iii) $\rho_B : B \to KB$ is an isomorphism in $\mathscr{B}$.

(c) Show that any reflective subcategory of a complete category is complete.

6

Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathscr{X}$.

(a) Show that the forgetful functor $G^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \to \mathscr{X}$ has a left adjoint $F^{\mathbb{T}}$, where $\mathscr{X}^{\mathbb{T}}$ is the category of $\mathbb{T}$-algebras.

(b) Given an adjunction $F \dashv G : \mathscr{A} \to \mathscr{X}$ inducing $\mathbb{T}$, show that there is a functor $K : \mathscr{A} \to \mathscr{X}^{\mathbb{T}}$ such that $G^{\mathbb{T}}K = G$ and $KF = F^{\mathbb{T}}$.

(c) Show that the following conditions are equivalent:

(i) $\mu : TT \to T$ is an isomorphism.

(ii) $T\eta = \eta T$.

(iii) For every $\mathbb{T}$-algebra (A, a), the morphism $a : TA \to A$ is an isomorphism in $\mathscr{X}$.

(iv) The forgetful functor $G^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \to \mathscr{X}$ is fully faithful.

END OF PAPER