UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 1:30 pm to 4:30 pm

PAPER 21

GEOMETRIC GROUP THEORY

There are ***FOUR*** *questions in total.*

You may attempt all ***FOUR*** *questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) What does it mean for a map of metric spaces $f : X \to Y$ to be a *quasi-isometric embedding*? What does it mean for f to be a *quasi-isometry*? When X and Y are finitely generated groups equipped with word metrics, explain briefly why these definitions are independent of the choices of word metrics.

(b) Let Γ be a finitely generated group and H a subgroup of finite index. Prove that the inclusion map $H \hookrightarrow \Gamma$ is a quasi-isometry.

(c) Let Γ be a finitely generated group. Recall that a *retraction* is a homomorphism $r : \Gamma \to H$ with a right-inverse $i : H \to \Gamma$. Show that if r is a retraction then H is also finitely generated and the right-inverse i is a quasi-isometric embedding.

(d) Prove that every subgroup of a finitely generated abelian group is quasi-isometrically embedded.

[You may find it helpful to appeal to the classification of finitely generated abelian groups.]

(e) Let X be a geodesic metric space. What does it mean to say that a subspace Y is *κ-quasiconvex* for some κ?

(f) Consider $\mathbb{Z}^2$ equipped with its standard generating set. Give an example of a subgroup H such that, in $\mathrm{Cay}(\mathbb{Z}^2)$, H is quasi-isometrically embedded, but not κ-quasiconvex for any κ.

2

(a) State and prove the Schwarz–Milnor Lemma.

(b) What does it mean to say that a geodesic metric space is δ-*hyperbolic* for some δ? What does it mean to say that a group is *hyperbolic*?

(c) Let X be a δ-hyperbolic metric space and let $c, c' : [a, b] \to X$ be geodesics with $c(a) = c'(a)$. Prove that

$$d(c(t), c'(t)) \leqslant 2\delta + d(c(b), c'(b))$$

for any $t \in [a, b]$.

(d) Let Γ be a group acting cocompactly and properly discontinuously by isometries on a proper, geodesic, δ-hyperbolic metric space X. Stating carefully any results from the course that you need, prove that Γ is hyperbolic and that the conjugacy problem is solvable in Γ.

3

(a) Let G be a group acting by isometries on a tree T without inversions. Explain how to recover a graph of groups with fundamental group G from the action of G on T.

[You do *not* need to show that your construction is independent of the choices you make.]

(b) Consider the group $G = \mathbb{Z}/2 * \mathbb{Z}/3$. Prove that every element of G with finite order is conjugate into either the obvious copy of $\mathbb{Z}/2$ or the obvious copy of $\mathbb{Z}/3$.

(c) By considering a homomorphism $G \to \mathbb{Z}/6$ or otherwise, find a subgroup K of finite index in G in which every non-trivial element has infinite order.

(d) Prove that K is free.

4

(a) State and prove Van Kampen's Lemma.

(b) Let $\mathcal{P} \equiv \langle a, b, c \mid [a, b], [b, c], [c, a] \rangle$, the standard presentation of $\mathbb{Z}^3$. Show from first principles that $\delta_{\mathcal{P}}(n) \succeq n^2$.

(c) Exhibit a reduced word $w \in \{a, b, c\}^*$, null-homotopic in $\mathbb{Z}^3$, that has two distinct Van Kampen diagrams of minimal area.

END OF PAPER