UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 1:30 pm to 4:30 pm

PAPER 20

TOPICS IN ALGEBRAIC GEOMETRY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Define a *morphism* $X \to Y$ between schemes.

(b) Let X be an arbitrary scheme, $Y = \operatorname{Spec} A$ an affine scheme. Show that the set of morphisms $X \to Y$ is in one-to-one correspondence with the set of ring homomorphisms $A \to \Gamma(X, \mathcal{O}_X)$.

(c) Let A be a local ring. Show that the set of morphisms $\operatorname{Spec} A \to \mathbb{P}^n_{\mathbb{Z}}$ is in one-to-one correspondence with the following set T. We consider the subset $S \subseteq A^{n+1}$ consisting of $(n+1)$-tuples of elements in A with at least one entry in $A^\times$, the group of units of A. Then T is the set of equivalence classes of S for the equivalence relation such that $(a_i) \sim (a_i')$ if and only if there exists $a \in A^\times$ such that $(aa_i) = (a_i')$.

State briefly what goes wrong with your argument if A is not a local ring?

2

(a) Define the notion of scheme-theoretic fibre of a morphism $f : X \to Y$.

(b) Describe the fibres of the following morphisms defined by the obvious maps of rings in each case.

$\operatorname{Spec} k[T,U]/(T^2 - U^2) \to \operatorname{Spec} k[T]$

$\operatorname{Spec} \mathbb{Z}[T]/(T^2 + 1) \to \operatorname{Spec} \mathbb{Z}$

$\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$

(c) Let S, X and Y be schemes, $f : X \to S$, $g : Y \to S$ be morphisms. Let $p : X \times_S Y \to X$, $q : X \times_S Y \to Y$ be the projections. Show that there exists a point $z \in X \times_S Y$ with $p(z) = x$, $q(z) = y$ if and only if $f(x) = g(y)$. [Hint: Use the fact that if $K \subseteq L, L'$ are two field extensions of K, then $L \otimes_K L'$ is non-zero.]

(d) Show that a morphism being surjective is stable under base-change.

3

(a) State Riemann-Roch for curves.

(b) Let C be a non-singular projective curve of genus 4, and assume C is not hyperelliptic. Show C can be embedded in $\mathbb{P}^3$ as a complete intersection of a quadric surface and cubic surface.

(c) Let $C \subseteq \mathbb{P}^n$ be a non-singular projective curve of degree 5, and assume that C is not contained in a plane. Show the genus of C is at most 2.

4

(a) Define the notion of closed immersion.

(b) Show that if $f : X \to Y$ and $g : Y \to Z$ are closed immersions, then so is $g \circ f$.

(c) Show that being a closed immersion is stable under base-change.

(d) Define the notion of a separated morphism.

(e) Let $f : X \to Y$, $g : Y \to Z$ be morphisms with g separated. Show that the morphism $\Gamma_f : X \to X \times_Z Y$ induced by the identity map $X \to X$ and $f : X \to Y$ is a closed immersion.

Suppose further that $g \circ f$ is a closed immersion. Show that f is also a closed immersion.

5

(a) Let X be a scheme over a field k, $\mathcal{L}$ a line bundle on X.

Say what it means for $\mathcal{L}$ to be generated by global sections.

If $\mathcal{L}$ is generated by global sections, sketch how to construct a morphism $X \to \mathbb{P}^n_k$.

If X is a projecive non-singular scheme, state a criterion for when such a morphism is a closed immersion.

(b) Let $C \subseteq \mathbb{P}^2$ be an irreducible curve of degree d. What is $\mathrm{Cl}(\mathbb{P}^2 \setminus C)$?

(c) Let $X \subseteq \mathbb{P}^3$ be the quadric surface given by $x_0x_1 - x_2x_3 = 0$. Consider the linear system $\mathfrak{d}$ on X given by

$$\mathfrak{d} = \{X \cap H \mid H \subseteq \mathbb{P}^3 \text{ a plane containing } P = (1,0,0,1)\}.$$

Show this linear system defines a morphism $\varphi : X \setminus \{P\} \to \mathbb{P}^2$. Describe the fibres of this morphism.

(d) Consider the curve $C \subseteq \mathbb{A}^2$ defined by the equation $x^2 - y^5 = 0$. Describe a sequence of blow-ups $X_n \to X_{n-1} \to \cdots \to X_1 = \mathbb{A}^2$ such that the proper transform of C in X_n has no singular point mapping to $0 \in \mathbb{A}^2$.

END OF PAPER