UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 9:00 am to 12:00 pm

PAPER 2

LIE ALGEBRAS AND THEIR REPRESENTATIONS

Attempt no more than ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

All Lie algebras are finite-dimensional. Unless specified otherwise, all Lie algebras are complex.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Triangular graph paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

What does it mean for an element $x \in \mathrm{End}(V)$ to be nilpotent? What does it mean for a Lie algebra to be nilpotent?

Let k be an arbitrary field. State and prove Engel's theorem for Lie algebras over k.

Prove, or give a counterexample to the following. There is a basis of V such that a given nilpotent Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(V)$ is represented by strictly upper-triangular matrices.

Deduce from Engel's theorem that (*) a Lie algebra $\mathfrak{g}$ is nilpotent if and only if for every $x \in \mathfrak{g}$ we have ad x is a nilpotent endomorphism $\mathfrak{g} \to \mathfrak{g}$.

Now let k be algebraically closed (or $k = \mathbb{C}$ if you prefer). Using (*) or otherwise, prove that a Lie algebra is nilpotent if and only if every two dimensional subalgebra is abelian.

2

Let V be a representation for $\mathfrak{g} = \mathfrak{sl}_2 = \langle X, H, Y \rangle$ and let $v \in V$ be a highest weight vector of weight m for $\mathfrak{g}$. Prove by induction that

$$(XY)Y^k v = (k+1)(m-k)Y^k v.$$

State and prove Freudenthal's multiplicity formula.

You may assume the existence and any properties of a Casimir operator, the Cartan decomposition, the Killing form, simple roots, fundamental dominant weights, subalgebras $\mathfrak{s}_\alpha = \langle X_\alpha, H_\alpha, Y_\alpha \rangle$.

3

Let V be a finite-dimensional vector space and let $X \in \mathfrak{gl}(V)$. Prove that there exist a diagonalisable element $X_s \in \mathfrak{gl}(V)$ and nilpotent element $X_n \in \mathfrak{gl}(V)$ which are unique subject to $X = X_s + X_n$ and $[X_s, X_n] = 0$.

With the above notation, prove $(\mathrm{ad}X)_s = \mathrm{ad}(X_s)$ and $(\mathrm{ad}X)_n = \mathrm{ad}X_n$.

Now let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a *semisimple* Lie algebra. Prove that for all $X \in \mathfrak{g}$, we have X_n and X_s also in $\mathfrak{g}$.

You may use Weyl's theorem on complete reducibility and Schur's lemma.

Give examples of non-semisimple subalgebras $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ where the implication

$$X \in \mathfrak{g} \;\Rightarrow\; (X_n \in \mathfrak{g} \text{ and } X_s \in \mathfrak{g})$$

(i) holds, and (ii) doesn't hold.

4

If V is a finite-dimensional representation of a Lie algebra $\mathfrak{g}$ over a field k with basis $\{v_1, \ldots, v_n\}$, then construct the representation $\bigwedge^2 V$ of $\mathfrak{g}$.

For any two finite-dimensional representations V and W of a Lie algebra $\mathfrak{g}$, prove that

$$\bigwedge^2 (V \oplus W) \cong \bigwedge^2 V \oplus \bigwedge^2 W \oplus (V \otimes W). \qquad (*)$$

Let $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{C})$ and let $V = \Gamma_{1,0}$ be the natural representation. Calculate the decomposition into irreducibles of

$$\bigwedge^2 \left((S^2 V) \otimes V^* \right).$$

(Hint: it is highly recommended to use (*) when possible.)

5

Define the notion of a Cartan subalgebra $\mathfrak{h}$ of a semisimple Lie algebra $\mathfrak{g}$ and indicate how it gives rise to a set of roots R of $\mathfrak{g}$.

Show that for every root α there is a subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ of $\mathfrak{g}$ with basis $\{X_\alpha, Y_\alpha, H_\alpha\}$, with $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{h}$, satisfying $[H_\alpha, X_\alpha] = 2X_\alpha$, $[H_\alpha, Y_\alpha] = -2Y_\alpha$ and $[X_\alpha, Y_\alpha] = H_\alpha$.

(You may assume that 'weights add' and any properties of the Killing form if stated clearly. You may use Lie's theorem without proof.)

State the axioms defining an abstract root system.

By verifying each axiom above, show that for the set of roots R of $\mathfrak{g}$, the subset $R \subseteq \mathbb{E} := \mathbb{R}R = \mathbb{Z}R \otimes_{\mathbb{Z}} \mathbb{R}$ is an abstract root system, where $\mathbb{E}$ is equipped with the Killing form.

(You may use any facts about the representation theory of $\mathfrak{sl}_2$ if stated clearly.)

6

Define a (real) Lie group G and its tangent space at the identity, $T_e(G)$.

Define the exponential and logarithm maps on G when $G = \mathrm{GL}_n$.

Assuming the existence of an exponential map $\exp : \mathfrak{g} := T_e(G) \to G$ satisfying $d\exp : \mathfrak{g} \to \mathfrak{g}$ is the identity map, use the implicit function theorem to define the define the Lie bracket and show it is anti-symmetric and bilinear.

Define the maps Ad and ad and show $\mathrm{ad}(X)Y = [X, Y]$. Prove the Jacobi identity.

You may assume the following facts:

1. *Given $f\colon G \to H$ a Lie group homomorphism and $X \in T_eG$, we have that $f(\exp X) = \exp(\mathsf{d}f_e X)$.*
2. *Using the exponential map, one can prove: given $f\colon G \to H$ a Lie group homomorphism with G connected, then f is completely determined by $\mathsf{d}f_e\colon T_eG \to T_eH$.*
3. *Let G and H be maps of Lie groups with G simply connected and let $\mathfrak{g}$ and $\mathfrak{h}$ be their Lie algebras. A linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ is the differential of a map $A : G \to H$ of Lie groups if and only if α is a map of Lie algebras.*

Show that a subgroup H of a Lie group G is normal implies that the corresponding Lie algebra $\mathfrak{h}$ is an ideal of $\mathfrak{g}$.

END OF PAPER