UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday, 3 June, 2015 9:00 am to 12:00 pm

PAPER 19

HOMOTOPY THEORY

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define what it means for a map $\pi : E \to B$ to be a *Serre fibration.* Let $e_0 \in E$, $b_0 := \pi(e_0)$, $F := \pi^{-1}(b_0)$, and B be path-connected. Describe the long exact sequence relating $\pi_*(F, e_0)$, $\pi_*(E, e_0)$, and $\pi_*(B, b_0)$, defining all the maps involved.

[You do not need to prove that this sequence is exact.]

Let E and B be CW-complexes, B be simply-connected, $\pi : E \to B$ be a Serre fibration with fibre a $K(G, n)$ for an abelian group G, and $H^{n+1}(B; G) = 0$. Show that E is weakly homotopy equivalent to $B \times K(G, n)$.

2 Compute the integral cohomology ring of ΩS^{2n+1} for $n \geqslant 1$, stating carefully any results which you use.

Concatenation of loops defines a continuous map

$$\mu : \Omega S^{2n+1} \times \Omega S^{2n+1} \longrightarrow \Omega S^{2n+1}.$$

Describe briefly how this may be used to make $H_*(\Omega S^{2n+1}; \mathbb{Z})$ into a ring, and compute a presentation for this ring.

3 State the Hurewicz theorem modulo the Serre class of finitely generated abelian groups.

If X is a simply-connected space for which $H_i(X; \mathbb{Z})$ is finitely generated for each $i \geqslant 0$, show that there is a CW-approximation $f : C \to X$ where C has finitely many cells of each dimension.

Suppose that in addition $H_i(X; \mathbb{Z}) = 0$ for all $i > n$. Show that there is a CW-approximation $f : C \to X$ where C is a finite CW-complex of dimension $(n + 1)$.

4 Compute $H^*(K(\mathbb{Z}, n); \mathbb{Q})$ as a ring for all $n \geqslant 1$. Hence compute $\pi_i(S^{2n+1}) \otimes \mathbb{Q}$ for all $n \geqslant 0$ and $i \geqslant 1$, and compute $\pi_i(\mathbb{CP}^2 \# \mathbb{CP}^2) \otimes \mathbb{Q}$ for all $i \geqslant 1$.

5 Compute $H^i(K(\mathbb{Z}/2, n); \mathbb{F}_2)$ for all $n \geqslant 2$ and $i \leqslant n + 3$. Describe your answer in terms of Steenrod squares.

[You may state the $\mathbb{F}_2$-cohomology ring of $K(\mathbb{Z}/2, 1)$.]

Hence compute $\pi_i(\mathbb{RP}^{10}/\mathbb{RP}^6)$ for $1 \leqslant i \leqslant 9$.

END OF PAPER