UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 4 June, 2015 1:30 pm to 4:30 pm

PAPER 18

COMPLEX MANIFOLDS

*Attempt no more than **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

1. Let X be a real manifold with $\dim_{\mathbb{R}} X = 2n$. Define what is meant by a *holomorphic atlas* on X and also what is meant by an *almost complex structure* on X. Prove that any holomorphic atlas on X induces a natural almost complex structure.

2. Now suppose that J is an arbitary almost complex structure on X. Explain how J defines the vector bundle $TX^{0,1}$ and $TX^{0,1}$. If $z_1, \ldots, z_n$ are local holomorphic coordinates on X define the quantities $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$ and prove that the sets

$$\left\{ \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n} \right\}$$

and

$$\left\{ \frac{\partial}{\partial \overline{z}_1}, \ldots, \frac{\partial}{\partial \overline{z}_n} \right\}$$

give local frames for $TX^{1,0}$ and $TX^{0,1}$ respectively.

3. For smooth vector fields α, β on X define

$$N(\alpha, \beta) = 2([J\alpha, J\beta] - [\alpha, \beta] - J[\alpha, J\beta] - J[J\alpha, \beta])$$

where $[\alpha, \beta] = \alpha\beta - \beta\alpha$ denotes the Lie bracket on vector fields. Prove that if J is the almost complex structure induced by a complex structure then $N = 0$.

4. Now write the decomposition of a smooth vector field α as $\alpha = \alpha' + \alpha''$ where $\alpha' \in C^\infty(X, TX^{1,0})$ and $\alpha'' \in C^\infty(X, TX^{0,1})$. Prove that

$$N(\alpha, \beta)'' = -8[\alpha', \beta']''$$

and show that $N = 0$ if and only if both $TX^{1,0}$ and $TX^{0,1}$ are involutive under the Lie bracket (i.e. if and only if whenever α and β are smooth sections of $T^{1,0}X$ then so is $[\alpha, \beta]$ and similarly for $T^{0,1}X$).

2

(a). Let X be a complex manifold. Define the sheaf $\mathcal{A}^k$ of k forms and the sheaf $\mathcal{A}^{p,q}$ of (p,q)-forms on X and show that

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X).$$

Finally define the Dolbeaut cohomology groups $H^{p,q}_{\overline{\partial}}(X)$.

(b). The *Bott-Chern* cohomology group of X is defined to be

$$H^{p,q}_{BC}(X) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) : d\alpha = 0\}}{\partial\overline{\partial}\mathcal{A}^{p-1,q-1}(X)}.$$

Explain how this is well defined, and prove that complex conjugation gives an identification $H^{p,q}_{BC}(X) = \overline{H^{q,p}_{BC}(X)}$.

(c). Prove that if B is a polydisk then $H^{p,q}_{BC}(B) = 0$ for all $p, q \geqslant 1$. Does this statement still hold if B is replaced by an arbitary complex manifold? Justify your answer.

[You may use the Poincaré Lemmas which states that if B is a polydisk then the de-Rham cohomology $H^k_{dR}(B) = 0$ for all $k \geqslant 1$ and $H^{p,q}_{\overline{\partial}}(B) = 0$ for all $q \geqslant 1$.]

(d). Consider next the map $\phi \colon H^{p,q}_{BC}(X) \to H^{p,q}_{\overline{\partial}}(X)$ which takes the class of $\alpha \in \mathcal{A}^{p,q}(X)$ in $H^{p,q}_{BC}(X)$ to its class in $H^{p,q}_{\overline{\partial}}(X)$. Prove that ϕ is well defined.

Now suppose that X is compact and Kähler. Prove that the map ϕ is surjective.

3

(a). Let $\mathcal{F}$ be a sheaf of abelian groups on a complex manifold X and $\mathcal{U}$ be a locally finite open cover of X. Define the group $C^p(\mathcal{U}, \mathcal{F})$ of *Cech cochains*, the *coboundary map*

$$\delta\colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}).$$

Using this define briefly the *Cech cohomology groups*

$$\check{H}^p(X, \mathcal{F}).$$

(b). Now suppose X is compact with $\dim X \geqslant 2$ and let

$$X_0 = X \setminus \{p_1, \ldots, p_r\}$$

where $p_1, \ldots, p_r$ are distinct points in X. Show that for any holomorphic vector bundle E on X there is an isomorphism

$$\check{H}^p(X, \mathcal{O}(E)) \simeq \check{H}^p(X_0, \mathcal{O}(E|_{X_0})) \text{ for all } p \geqslant 0$$

where $\mathcal{O}(E)$ denotes the sheaf of holomorphic sections of E, and $E|_{X_0}$ denotes the restriction of E to X_0.

Does this continue to be true if $\dim X = 1$? Justify your answer.

[*You may assume the following version of Hartog's Theorem without proof: any holomorphic function* $f\colon B \setminus \{0\} \to \mathbb{C}$ *where* B *is the unit ball in* $\mathbb{C}^n$ *with* $n \geqslant 2$ *extends to a holomorphic* $\overline{f}\colon B \to \mathbb{C}$].

(c). Consider next $X = \mathbb{P}^1$. Define the *tautological line bundle* $\mathcal{O}_{\mathbb{P}^1}(-1)$ as well as the line bundles $\mathcal{O}_{\mathbb{P}^1}(k)$ for all $k \in \mathbb{Z}$.

Now suppose $[z, w]$ are homogeneous coordinates on $\mathbb{P}^1$, and let $\mathcal{U} = \{U_0, U_1\}$ where $U_0 = \{[1, w] : w \in \mathbb{C}\}$ and $U_1 = \{[z, 1] : z \in \mathbb{C}\}$. Using your definition of $\mathcal{O}_{\mathbb{P}^1}(-1)$, write down trivialisations of $\mathcal{O}_{\mathbb{P}^1}(-1)|_{U_0}$ and $\mathcal{O}_{\mathbb{P}^1}(-1)|_{U_1}$ and compute the associated transition function. Using this, identify the groups

$$C^q := C^q(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(k))$$

for $q = 0, 1$ and all $k \geqslant 0$ as well as the coboundary map $\delta\colon C^0 \to C^1$. Finally use this to compute

$$\dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k)) \text{ for all } k \geqslant 0.$$

4

(a). Let E be a holomorphic vector bundle on a Kähler manifold X and let h be a hermitian metric on E. Define the *Chern connection* ∇_h associated to h and prove that it exists and is unique. Define also the curvature form F_h of ∇_h explaining how it may be considered as an element of $\mathcal{A}^2(End(E))$.

(b). Explain how ∇_h induces operators

$$\nabla_h' \colon \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p+1,q}(E) \text{ and } \nabla_h'' \colon \mathcal{A}^{p,q}(E) \to \mathcal{A}^{p,q+1}(E)$$

and define the two associated Laplacian operators Δ' and Δ''. Prove the identity

$$\Delta'' = \Delta' + [iF_h, \Lambda]$$

where Λ denotes the dual of the Lefschetz operator L.

[*You may assume the identities* $[\Lambda, \nabla_h''] = -i(\nabla_h')^*$ *and* $[\Lambda, \nabla_h'] = i(\nabla_h'')^*$]

(c). Now assume that X is compact. Define what it means for a holomorphic line bundle F on X to be *positive*. Show that if F is a positive holomorphic line bundle and E is any holomorphic vector bundle then there exists an m_0 such that for all $m \geqslant m_0$ we have

$$H^0(X, E \otimes F^{-m}) = 0$$

where $F^{-m} = (F^*)^{\otimes m}$.

[*You may assume, if needed, that* $[L, \Lambda] = (p + q - n)\,\mathrm{Id}$ *on* $\mathcal{A}^{p,q}$].

(d). Is it true that in the above one can choose m_0 that works uniformly over all holomorphic vector bundles E? Justify your answer.

5

(a). Let E be a holomorphic vector bundle on a compact complex manifold X. Define the sheaf $\mathcal{A}^{p,q}(E)$ of (p,q)-forms with values in E, and state Dolbeaut's Theorem for (p,q)-forms with values in E.

(b). Assume now ω is a Kähler form on X and h is a hermitian metric on E. State the Hodge decomposition Theorem for (p,q)-forms with values in E, defining clearly the spaces involved.

(c). Suppose now L is a holomorphic line bundle on X and h is a positive hermitian metric on L such that $iF_h = \omega$ where F_h denotes the curvature of the Chern connection associated to h. The Spectral Gap Theorem states that there exists a k_0 and $\epsilon > 0$ such that for all $k \geqslant k_0$ and $q \geqslant 1$ the lowest eigenvalue of $\Delta_{\overline{\partial}_{L^k}} : \mathcal{A}^{0,q}(X, L^k) \to \mathcal{A}^{0,q}(X, L^k)$ is bounded below by ϵk.

Explain how to deduce from this that

$$H^q(X, L^k) = 0 \text{ for } q \geqslant 1 \text{ and } k \geqslant k_0.$$

(d). Finally suppose X is a compact complex manifold with $\dim X = 1$ and L is a positive holomorphic line bundle on X. Suppose z is a local coordinate on X defined on a coordinate chart U centered at $x_0 \in X$ and let ζ be a holomorphic frame for L defined on U. Prove that there exists a k_0 such that for any $k \geqslant k_0$ and any choice of $a_{-1}, \ldots, a_{-r} \in \mathbb{C}$, there exists an $s_k \in H^0(X \setminus \{x_0\}, L^k)$ which locally near x_0 can be written as

$$s_k(z) = \left(\sum_{j=-r}^{-1} a_j z^j + \sum_{j \geqslant 0} a_{jk} z^j \right) \zeta^k \text{ for } z \neq 0$$

where $a_{jk} \in \mathbb{C}$ for $j \geqslant 0$.

END OF PAPER