UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 1:30 pm to 4:30 pm

PAPER 17

DIFFERENTIAL GEOMETRY

*Attempt no more than **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define what is meant by a *tangent vector* at a point p in a manifold M. Explain why the set of all tangent vectors at p is a vector space of dimension equal to $n = \dim M$. Define the *cotangent space* T_p^*M and state the transformation law for coordinates of an element $\alpha \in T_p^*M$ corresponding to a change of local coordinates on M around p.

Define the *cotangent bundle* T^*M. Construct a family of charts making T^*M into a manifold and a vector bundle over M.

Let a_i denote the coordinates on the cotangent spaces with respect to a basis dx_i, where x_i are local coordinates on M. Show that the 2-forms $\sum_{i=1}^n da_i \wedge dx_i$ are independent of the choice of local coordinates and are local expressions for some 2-form η which is well-defined on all of T^*M. Show that the manifold T^*M is always orientable.

[*Basic results about orientation on manifolds may be assumed if accurately stated.*]

2

Define what is meant by a *vector field* on a manifold M and by the *Lie bracket* $[X, Y]$ of vector fields X, Y on M. (If your definition of $[X, Y]$ uses a choice of local coordinates, then you should show that the Lie bracket is independent of that choice.)

Let G be a Lie group; what does it mean to say that a vector field X on G is *left-invariant*? Show that the left-invariant vector fields on G form a vector space isomorphic to the tangent space $\mathfrak{g} = T_I G$ at the identity element and that the Lie bracket of two left-invariant vector fields is left-invariant.

Now let $G = SO(n)$ be a special orthogonal group. Explain briefly how $\mathfrak{so}(n) = T_I SO(n)$ may be identified with the space of all skew-symmetric $n \times n$ real matrices. Show that the Lie bracket of left-invariant vector fields induces a Lie bracket on $\mathfrak{so}(n)$ given by $[B_1, B_2] = B_1 B_2 - B_2 B_1$.

[*You may assume that $SO(n)$ is an embedded submanifold of $GL(n, \mathbb{R})$.*]

3

Define a *covariant derivative* ∇ on a vector bundle $\pi : E \to M$. Explain what is meant by the covariant derivative ∇_A induced by a connection A on E. Prove that every covariant derivative on E is induced by some connection.

Let $\gamma : I \to M$ be an embedded curve (i.e. a 1-dimensional embedded submanifold), where $I \subset \mathbb{R}$ is an open interval and $[0,1] \subset I$. Define what is meant by a *horizontal lift* $\tilde{\gamma}$ of γ with respect to A. Show that for each $a_0 \in E$ with $\pi(a_0) = \gamma(0)$ there exists unique horizontal lift of γ satisfying $\tilde{\gamma}(0) = a_0$ and the assignment $\tilde{\gamma}(0) \mapsto \tilde{\gamma}(1)$ defines a linear isomorphism $E_{\gamma(0)} \to E_{\gamma(1)}$ between respective fibres of E. (You should state clearly any standard results you require about ordinary differential equations.)

Now suppose that E is endowed with an inner product $\langle \cdot, \cdot \rangle$ on the fibres (smoothly varying with the fibres), such that

$$X\langle s_1, s_2 \rangle = \langle \nabla^A_X s_1, s_2 \rangle + \langle s_1, \nabla^A_X s_2 \rangle, \qquad (*)$$

for all vector fields X on M and all sections s_1, s_2 of E. Here $\nabla^A_X s$ denotes a section of E obtained by the bilinear pairing of the 1-form $\nabla^A s$ with a vector field X. Prove that then for all embedded curves $\gamma : I \to M$ the linear maps $E_{\gamma(0)} \to E_{\gamma(1)}$ defined above are isometries.

[*You may assume that a smooth section of E or TM along γ can be extended to a smooth section over some open neighbourhood of $\gamma(I)$ in M.*]

4

Let (M, g) be a Riemannian manifold. State the ordinary differential equations satisfied by geodesics on M in local coordinates, defining clearly all terms appearing in the equation. Define what is meant by the *exponential map* at $p \in M$ and prove that the exponential map induces well-defined local coordinates (the geodesic coordinates) on some neighbourhood of p.

Prove that the coefficients Γ^i_{jk} of the Levi–Civita connection D of g written in the geodesic coordinates at p vanish at p.

What is a *geodesic sphere* around p? State and prove Gauss' lemma.

[*You may assume that solutions of the geodesic ODEs are parametrized with constant speed. Basic properties of symmetric connections on M may be used without proof if accurately stated.*]

5

Let (M, g) be an oriented Riemannian manifold. Define the inner product induced by g on the fibres of $\Lambda^p T^* M$. Define the *Hodge $*$ operator* for M (you are not expected to define the volume form of g) and compute the square of $*$ on differential p-forms.

Deduce from Stokes' theorem the expression for the formal (L^2) adjoint δ of the exterior derivative in terms of $*$ and d. Define the *harmonic forms* on M and show that if M is compact then every harmonic form α satisfies $d\alpha = 0$ and $\delta\alpha = 0$.

Now let N be a 4-dimensional compact oriented Riemannian manifold. Show that every $\alpha \in \Omega^2(N)$ may be written as $\alpha = \alpha_+ + \alpha_-$, for some unique $\alpha_\pm$ satisfying $*\alpha_\pm = \pm\alpha$ (such $\alpha_\pm$ are called, respectively, self-dual and anti-self-dual 2-forms). Show that the expression $\int_N \alpha \wedge \beta$, for closed $\alpha, \beta \in \Omega^2(N)$, induces a non-degenerate symmetric bilinear form of signature $(b^+(N), b^-(N))$ on the de Rham cohomology $H^2_{\mathrm{dR}}(N)$, where $b^\pm(N) = \dim \mathcal{H}^\pm(N)$ and $\mathcal{H}^\pm(N)$ is the space of harmonic (anti-)self-dual forms on N.

State the Hodge decomposition theorem. Show that every exact 3-form on N can be expressed as the exterior derivative of a self-dual 2-form.

[*You may assume that every de Rham cohomology class on N is represented by a unique harmonic form.*]

END OF PAPER