UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday, 28 May, 2015 1:30 pm to 4:30 pm

PAPER 16

ALGEBRAIC GEOMETRY

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS

Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS

None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\theta : R \to A$ be a homomorphism of commutative rings and M and N be A-modules; show that there is a natural map of R-modules (and indeed A-modules) $M \otimes_R N \to M \otimes_A N$.

Let $\phi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of varieties, let $\mathcal{F}$, $\mathcal{G}$ be arbitrary $\mathcal{O}_X$-modules and let $\mathcal{H}$ be an $\mathcal{O}_Y$-module. Describe in detail the constructions of

(i) the $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$,

(ii) the $\mathcal{O}_Y$-module $\phi_*\mathcal{F}$, and

(iii) the $\mathcal{O}_X$-module $\phi^*\mathcal{H}$.

Exhibit a canonical morphism of $\mathcal{O}_Y$-modules $\mathcal{H} \to \phi_*\phi^*\mathcal{H}$, and interpret this in the case when $\mathcal{H} = \mathcal{O}_Y$.

By constructing a certain morphism of $\mathcal{O}_Y$-modules

$$\phi_*\mathcal{F} \otimes_{\mathcal{O}_Y} \phi_*\mathcal{G} \to \phi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

for any $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, deduce the existence of a morphism of $\mathcal{O}_Y$-modules (for any $\mathcal{O}_X$-module $\mathcal{F}$ and $\mathcal{O}_Y$-module $\mathcal{H}$)

$$\phi_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{H} \to \phi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \phi^*\mathcal{H}),$$

which is an isomorphism whenever $\mathcal{H}$ is locally free of finite rank.

[*The construction of the sheafification of a presheaf, and its properties, may be assumed in this question. You should not however assume that the sheaves are quasi-coherent.*]

2

Let $(X, \mathcal{O}_X)$ denote an algebraic variety over an algebraically closed field k. We assume that the variety X has the property that $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I} \subseteq \mathcal{O}_X$, and we let A denote the k-algebra $H^0(X, \mathcal{O}_X)$.

(a) If $r > 0$ and $\mathcal{F} \subset \mathcal{O}_X^r$ a coherent $\mathcal{O}_X$-submodule, prove, by induction on r or otherwise, that $H^1(X, \mathcal{F}) = 0$.

(b) If $P \in X$ has an open affine neighbourhood $U \ni P$ with $Y := X \setminus U$, prove that there exists $f \in A$ such that $Y \subseteq V(f) \subseteq X$, where $X_f := \{x \in X \ : \ f(x) \neq 0\}$ is affine and $f(P) = 1$.

(c) By repeated use of (b), we can find $f_1, \ldots, f_r \in A$ such that X_{f_i} is affine for each i and $X = \bigcup_{i=1}^r X_{f_i}$; prove that there exist $g_1, \ldots, g_r \in A$ such that $\sum_{i=1}^r g_i f_i = 1$. [Hint: *Use result from (a).*]

(d) Deduce from (c) that there exists a subring $B \subseteq A$, containing the f_i and g_i, such that B is a finitely generated k-algebra and $B_{f_i} = A_{f_i}$ for all i. For any $N > 0$, show that the ideal $\langle f_1^N, \ldots, f_r^N \rangle$ in B is the whole ring. Deduce further that $A = B$ and hence A is a finitely generated (reduced) k-algebra.

(e) Let Y denote the affine variety with coordinate ring A. By defining a suitable isomorphisn $\phi : X \to Y$, show that X is an affine variety.

If X now is an affine variety and $\mathcal{I} \subset \mathcal{O}_X$ a coherent sheaf of ideals, explain briefly why the converse to the above result holds, namely that $H^1(X, \mathcal{I}) = 0$.

If $X = \mathbf{P}^n \setminus Z$ where each component of the closed subvariety Z has codimension at least two; exhibit an explicit sheaf of ideals $\mathcal{I}$ for which $H^1(X, \mathcal{I}) \neq 0$.

[*Standard properties of sheaf cohomology may be assumed in this question.*]

3

For $\mathcal{F}$ a sheaf of abelian groups on a topological space X and $\mathcal{U} = \{U_1, \ldots, U_d\}$ an open cover of X, describe the construction of the Čech cohomology groups $\check{H}^i(\mathcal{U}, \mathcal{F})$ and prove that $\check{H}^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$. If X is a variety and $\mathcal{F}$ a quasi-coherent sheaf, state a condition on $\mathcal{U}$ for $\check{H}^i(\mathcal{U}, \mathcal{F})$ to be isomorphic to the sheaf cohomology group $H^i(X, \mathcal{F})$ for all $i \geqslant 0$.

For a variety $(X, \mathcal{O}_X)$, consider the multiplicative sheaf of units $\mathcal{O}_X^*$ in the structure sheaf. Show that the subgroup $\mathrm{Pic}(X)_{\mathcal{U}}$ of the Picard group $\mathrm{Pic}(X)$, consisting of isomorphism classes of invertible sheaves which are trivialized with respect to the open cover $\mathcal{U}$, is isomorphic to $\check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$. [*You may assume that the isomorphism class of an invertible sheaf is determined by giving its transition functions with respect to an open cover.*]

Now let V be an irreducible variety and $\mathcal{K}^*$ the (constant) multiplicative sheaf of non-zero rational functions — so $H^0(U, \mathcal{K}^*) = k(V)^*$ for any non-empty open set U. Prove that there is a natural map $k(V)^* \to H^0(V, \mathcal{K}^*/\mathcal{O}_V^*)$, whose cokernel is isomorphic to $\mathrm{Pic}(V)$. Quoting any results on sheaf cohomology that you may need, deduce that $\mathrm{Pic}(V) \cong H^1(V, \mathcal{O}_V^*)$.

Let $V \subset \mathbf{P}^3$ be an irreducible smooth quadric surface, whose curve at infinity C is a smooth conic, and let $U = V \setminus C$ be the corresponding smooth affine quadric surface; prove that $\mathrm{Pic}(U)$ is non-trivial.

4

For V an irreducible variety, describe the construction of the coherent $\mathcal{O}_V$-module of Kähler forms Ω^1_V. For $P \in V$, define the *Zariski tangent space* $T_{V,P}$ and show that $T_{V,P}$ is the dual space of $\Omega^1_{V,P}/m_P\Omega^1_{V,P}$, where $\Omega^1_{V,P}$ denotes the stalk of Ω^1_V at P and $m_P \subset \mathcal{O}_{V,P}$ is the maximal ideal of the local ring $\mathcal{O}_{V,P}$.

Define what is meant by $P \in V$ being a *smooth* point of V. Assuming the result that the smooth locus is open and dense, state and prove the *Jacobian criterion* (in terms of a matrix of partial derivatives evaluated at P) for a point on an affine variety $V \subseteq \mathbf{A}^N$ to be smooth. For V any irreducible smooth variety, show that Ω^1_V is a locally free $\mathcal{O}_V$-module of rank n, where $n = \dim V$.

Suppose $W \subseteq V$ are arbitrary irreducible varieties with W a non-empty closed subvariety of V, and let $\mathcal{I}_W \subset \mathcal{O}_V$ denote the corresponding sheaf of ideals. If $\iota : W \hookrightarrow V$ denotes the inclusion morphism, and $\mathcal{M}$ an $\mathcal{O}_V$-module on V, let $\mathcal{M}|_W$ denote the $\mathcal{O}_W$-module $\iota^*\mathcal{M}$. For $\mathcal{M}$ a quasi-coherent $\mathcal{O}_V$-module, give (without proof) an explicit description of $\mathcal{M}|_W$ on affine pieces. Show that there is an exact sequence of sheaves on W

$$\mathcal{I}_W/\mathcal{I}_W^2 \to \Omega^1_V|_W \to \Omega^1_W \to 0.$$

In the case when W is locally principal and not contained in the singular locus of V, show that the sheaf $\mathcal{I}_W/\mathcal{I}_W^2$ is an invertible $\mathcal{O}_W$-module and the morphism $\mathcal{I}_W/\mathcal{I}_W^2 \to \Omega^1_V|_W$ is injective.

[*Standard results about quasi-coherent sheaves on affine varieties may be assumed in this question.*]

END OF PAPER