UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday, 2 June, 2015 1:30 pm to 4:30 pm

PAPER 15

ALGEBRAIC TOPOLOGY

Attempt no more than ***FOUR*** *questions.*

There are ***FIVE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the degree of a map $f : S^n \to S^n$. If p is a regular point in the image of f, explain how $\deg f$ is related to the set $f^{-1}(p)$. (If you want, you may assume f is differentiable.)

If $F : (D^{n+1}, S^n) \to (D^{n+1}, S^n)$, let $\widetilde{F} : D^{n+1}/S^n \to D^{n+1}/S^n$ be the map induced by F. Show that $\deg \widetilde{F} = \deg F|_{S^n}$.

Let $\Gamma = \{(x, F(x)) \mid x \in D^{n+1}\} \subset D^{n+1} \times D^{n+1}$ be the graph of F. Show that if Γ is transverse to $N = D^{n+1} \times \{0\}$, then $\deg F|_{S^n}$ is the number of points in $N \cap \Gamma$, counted with appropriate signs.

2 Suppose C and C' are free finitely generated chain complexes over a commutative ring R. Define the tensor product $C \otimes C'$. (You may take as given the operation of tensor product of R-modules.)

A R-linear map $f : C \to C'$ is said to shift grading by j if $f : C_i \to C'_{i+j}$ for all $i \in \mathbb{Z}$. Let $M_j(C, C')$ be the set of all such maps, and define $d : M_j(C, C') \to M_{j-1}(C, C')$ by $d(f) = f \circ d_C + (-1)^{j-1} d_{C'} \circ f$. Show that $d^2 = 0$, and that there is a natural bijection between the homology group $H_0(M(C, C'))$ and the set of chain homotopy classes of chain maps $f : C \to C'$.

Show further that $M_*(C, C') \simeq X \otimes C'$ for some chain complex X which you should specify. (*Hint:* conjugate the differential on $X \otimes C'$ by an automorphism of $X \otimes C'$ which acts as multiplication by $(-1)^{\rho(j)}$ on $X_i \otimes C'_j$, where the function $\rho(j)$ is to be determined.)

3 The linear map $f : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ given by $f(x) = -x$ satisfies $f(\mathbb{Z}^{n-1}) = \mathbb{Z}^{n-1}$, so it descends to a map $F : T^{n-1} \to T^{n-1}$ on the quotient $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1} = T^{n-1}$. Let $M_n = T^{n-1} \times [0, 1]/\sim$, where $(x, 0) \sim (F(x), 1)$. Compute $H^*(M_4; \mathbb{Z})$.

Show that $H^*(M_n; \mathbb{Z}/2) \simeq H^*(T^n; \mathbb{Z}/2)$ as groups. By considering the intersection pairing on $H_1(M_2; \mathbb{Z}/2)$, determine the ring structure of $H^*(M_2; \mathbb{Z}/2)$. Are $H^*(M_n, \mathbb{Z}/2)$ and $H^*(T^n, \mathbb{Z}/2)$ isomorphic as rings? Justify your answer.

4 Show that $\mathbb{RP}^n$ can be given the structure of a cell complex with one cell of each dimension between 0 and n. What are the attaching maps? Write down the cellular chain complex associated to this cell decomposition, and explain how to compute the differentials in it.

Using Poincare duality (or otherwise), determine the structure of $H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ as a ring.

Compute the group $H^*(\mathbb{RP}^2 \times \mathbb{RP}^3 \times \mathbb{RP}^4)$. (You do not need to describe the ring structure.)

5 Suppose that S is a closed oriented surface smoothly embedded in a closed oriented 4-manifold M. Show that $\langle e(\nu_S), [S] \rangle = ([S] \cdot [S])$, where ν_S is the normal bundle of S in M.

Now suppose $M = \mathbb{CP}^2$, and that S is the image of the embedding $i : \mathbb{CP}^1 \to \mathbb{CP}^2$ given by $i([u : v]) = [u^2 + v^2 : uv : u^2 - v^2]$. What is $[S] \cdot [S]$? Use it to compute $H_*(\partial \nu(S))$, where $\nu(S)$ is denotes a closed tubular neighborhood of S. Compute $H_*(\mathbb{CP}^2 - \nu(S))$.

END OF PAPER