UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 5 June, 2015 1:30 pm to 4:30 pm

PAPER 14

TOPICS IN ERGODIC THEORY

*Attempt **BOTH** questions from Section I.*

*Attempt **ONE** question from Section II.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

Define ergodicity of a measure preserving system. Give examples of ergodic and non-ergodic systems. Prove your claims. (You may use without proof any criterion for ergodicity that was mentioned in the lectures or in the example sheets.)

State the von Neumann mean ergodic theorem. Prove the theorem in the special case, when the system is invertible. Identify the limit when the system is ergodic. Prove your claim.

State the Birkhoff pointwise ergodic theorem.

What is unique ergodicity? How can you sharpen the pointwise ergodic theorem for uniquely ergodic systems? (State a theorem without proof.)

2

Define weak mixing measure preserving systems.

Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system such that $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu, T \times T)$ is ergodic. Prove that $(X, \mathcal{B}, \mu, T)$ is weak mixing. (You may use without proof any auxiliary results that was used in the proof in the lectures.)

Prove that if $(X, \mathcal{B}, \mu, T)$ is a weak mixing system then

$$\text{C-lim}_{n \to \infty} \int f_0 U_T^n f_1 \cdots U_T^{nk} f_k d\mu = \int f_0 d\mu \cdots \int f_k d\mu$$

holds for all real valued $f_0, f_1, \ldots, f_k \in L^\infty(X)$. (You may use without proof any auxiliary results that was used in the proof in the lectures.)

Prove that the above result holds also if we exchange C-lim for D-lim.

SECTION II

3

Define conditional measures. (You need to state a theorem, but you do not need to prove it.)

Define the ergodic components of a measure preserving system. Prove that (almost all of) these measures are indeed ergodic. (You do not need to prove that they are invariant and you may use without proof any auxiliary results that were used in the proof in the lectures.)

What are the ergodic components of the circle rotation $(\mathbb{R}/\mathbb{Z}, \mathcal{B}, m, R_{1/5})$? Prove your claim.

4

Define almost periodic functions, and almost periodic functions with respect to a factor map. Define compact extensions.

Give an example of a factor map between two measure preserving systems and a function that is not almost periodic but almost periodic with respect to the factor map. Prove your claims.

Let $(X, \mathcal{B}_X, \mu, T)$ and $(Y, \mathcal{B}_Y, \nu, S)$ be two invertible ergodic measure preserving systems on Borel probability spaces and let $\varphi : X' \to Y$ be a compact extension. Prove that for every $A \in \mathcal{B}_X$ with $\mu(A) > 0$, there is a measurable set $B \subset A$ with $\mu(B) > 0$ such that χ_B is almost periodic with respect to φ.

END OF PAPER