UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 8 June, 2015 9:00 am to 11:00 am

PAPER 13

PROBABILISTIC COMBINATORICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State two equivalent forms of the Lusternik–Shnirelman–Borsuk theorem.

(ii) For $n = 2k + d \geqslant 2k \geqslant 2$ define the Kneser graph $\mathrm{KG}(n,k)$. Deduce from (i) the theorem of Lovász that the chromatic number of $\mathrm{KG}(n,k)$ is $d+2$.

(iii) Give an entirely combinatorial proof of the fact that $\chi\big(\mathrm{KG}(n,2)\big) = n-2$ for $n \geqslant 4$.

2

Let $X = (X_1, \ldots, X_n)$ be a sequence of random variables taking finitely many values. For $A \subset [n]$ set $X_A = (X_i)_{i \in A}$ and let $H(X_A)$ be the information theoretic entropy of X_A.

(i) Show that the map $\mathcal{P}(n) \to \mathbb{R}$, $A \mapsto H(X_A)$, is a submodular function on $\mathcal{P}(n)$.

(ii) Let $\mathcal{A}$ and $\mathcal{B}$ be finite multisets of subsets of $[n]$ with $\mathcal{A} > \mathcal{B}$, i.e. with $\mathcal{B}$ a compression of $\mathcal{A}$. Show that

$$\sum_{A \in \mathcal{A}} H(X_A) \geqslant \sum_{B \in \mathcal{B}} H(X_B).$$

(iii) Let $\mathcal{F}$ be a family of graphs on $[n]$ such that if $F, G \in \mathcal{F}$ then the graph $F \cap G$ has no isolated vertices. Show that

$$|\mathcal{F}| \leqslant 2^{n^2/2-n}.$$

3

Let $G_{n,1/2}$ be the random graph on $[n]$ with edge probability $1/2$. Prove the following assertions.

(i) Whp we have $\chi\big(G_{n,p}\big) \geqslant n/(2\log_2 n)$.

(ii) Let $X_r = X_r(G_{n,p})$ be the number of K_r subgraphs in $G_{n,1/2}$, and let r_0 be the maximal r such that

$$\mathbb{E}\big(X_r(G_{n,1/2})\big) \geqslant n^{9/5}.$$

Then $r_0 = \big(2 + o(1)\big)\log_2 n$.

(iii) Whp we have $\chi\big(G_{n,p}\big) = \big(1 + o(1)\big)n/(2\log_2 n)$.

State precisely the martingale inequalities you use.

4

The *multiplicative energy* $E(A,B)$ of two non-trivial finite sets of positive reals, A and B, is the cardinality of the set

$$\{(a,b,c,d) \in A \times B \times A \times B : \ a/b = c/d\}.$$

Show that

$$E(A,B) \geqslant \frac{|A|^2\,|B|^2}{|A \cdot B|}$$

and deduce that

$$\frac{|A|^2|B|^2}{4\lceil \log|B| \rceil} \leqslant |A \cdot B||A+A||B+B|.$$

END OF PAPER