UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 29 May, 2015 9:00 am to 11:00 am

PAPER 12

TECHNIQUES IN COMBINATORICS

*There are **SIX** questions in total.*

You may attempt any number of questions.
*Full marks can be gained from complete answers to **FOUR** questions.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) State and prove Chebyshev's inequality.

(ii) Let $p : \mathbb{N} \to [0,1]$ be a function such that $np(n) \to \infty$ as $n \to \infty$. Prove that if G is a random graph with n vertices, where the edges are chosen independently, each with probability $p(n)$, then the probability that G contains at least $2p(n)^3\binom{n}{3}$ triangles tends to zero as n tends to infinity.

2 Let A be a subset of $\mathbb{Z}_n$ of density α. Let $K = \{r : |\hat{A}(r)| \geqslant \theta\}$ and let B be the Bohr set $B(K;\epsilon) = \{x : \forall r \in K \ |1 - \omega^{rx}| \leqslant \epsilon\}$, where $\omega = \exp(2\pi i/n)$.

(i) Let $f = A * A$, let $u \in \mathbb{Z}_n$ and let $g(x) = f(x-u)$ for every $x \in \mathbb{Z}_n$. Find expressions for the Fourier transforms of f and g in terms of $\hat{A}$.

(ii) Prove that $\|\hat{A}\|_4^4 \leqslant \alpha^3$.

(iii) Prove that if $u \in B$, then $\|f - g\|_2^2 \leqslant \epsilon^2\alpha^3 + 4\theta^2\alpha$.

3 State and prove Plünnecke's inequality.

4 Assuming Szemerédi's regularity lemma and any facts you like about quasirandom graphs, prove that for every $\delta > 0$ there exists n such that for every subset $A \subset [n]^2$ of size at least δn^2 there exist x, y, d such that $d \neq 0$ and the points (x, y), $(x, y+d)$ and $(x+d, y)$ all belong to A.

5 Prove that if $A \subset \mathbb{F}_p^n$ contains a line in every direction, then $|A| \geqslant p^n/n!$.

6 Let G be a bipartite graph with vertex sets X and Y, both of size n. Suppose that every vertex of G has degree δn and that the 4-cycle density of G is at most $\delta^4(1 + c^4)$. Let $A \subset X$ and $B \subset Y$ be subsets of density α and β, respectively. Prove that the number of edges from A to B differs from $\delta|A||B|$ by at most $c\delta(\alpha\beta)^{1/2}n^2$. [If you wish to use facts about the 4-cycle norm, then you should prove them first.]

END OF PAPER